ELSEVIER Theoretical Computer Science 217 (1999) 407–436

Theoretical
Computer Science

Latticial structures in data analysis

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Abstract

In general, lattice theory has helped to simplify, unify and generalize many aspects of mathematics, and it has suggested many interesting new problems. In an era of ever-increasing proliferation of scientific results, its relative simplicity and unifying influence are certainly healthy and refreshing. (Birkhoff [4, p. 1])

Diagrams prove nothing, but bring outstanding features readily to the eye; they are therefore no substitute for such critical tests as may be applied to the data, but are valuable in suggesting such tests, and explaining the conclusions founded upon them. (Fisher [16, p. 24])

Quel que soit le point de départ de l'activité scientifique, cette activité ne peut pleinement convaincre qu'en quittant le domaine de base: si elle expérimente, il faut raisonner; si elle raisonne, il faut expérimenter. (Bachelard [41, p. 3])[1] (1)

Keywords: Galois lattices; Dualities; Data analysis

1. Introduction

A quarter of a century after the ebbing of the structuralist wave, which had reached many a scientific methodology before it departed, leaving in its wake a vanishing of formalized models, a new interest is emerging in latticial and other ordinal structures because of their ability to encode any kind of duality, in experimental as well as observational data. While my claim is not a complete novelty (see [1, 18, 20, 38, 49] and some personal contributions numbered [55–66] in references), it has nonetheless been bolstered by the ongoing development, since 1983, of the computer program *GLAD* (General Lattice Analysis and Design, see [66]). What I propose here is a kind of rehabilitation thesis, since from its birth as an autonomous mathematical topic during the thirties, *Lattice Theory* received enthusiastic reactions. Long before the revival of

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[1] The original French quotations are roughly translated in English at the end of the text.

PII: S0304-3975(98)00279-5

Galois Lattices in the sixties, a French philosopher of Mathematics as A. Lautman immediately recognized the interest of lattices within the scope of Mathematics and methodology of sciences, for encoding dualities:

> S'inspirant de Dedekind, un grand nombre de chercheurs contemporains parmi lesquels MM. Birkhoff, von Neumann, Glivenko, Ore et d'autres ont construit une théorie générale des structures (les auteurs anglais disent lattices, réseaux) qui englobe la théorie des ensembles, la théorie des nombres, la géométrie projective, la topologie combinatoire, la théorie des probabilités, la logique mathématique, la théorie des espaces fonctionnels, etc...(...) La théorie générale des structures repose donc sur la possibilité de structurer de deux façons inverses l'une de l'autre un même ensemble, et c'est pour nous un résultat d'une importance philosophique capitale de voir cette dualité interne de deux êtres antisymétriques, distinguables au sein d'un même être, former le principe générateur d'une immense moisson de réalité mathématique. (Lautman [49, p. 249] (2)

This peculiar welcome was reinforced by the existence, in these days, of a very active French lattice school around the work of Dubreil [12], Dubreil-Jacotin et al. [14] on *geometric lattices*, the papers by Riguet 1948 on *binary relations*, and Schützenberger [31]. Later, however, local reactions became more ambivalent. Although he wrote an enthusiastic foreword for a new edition of Lautman's work, J. Dieudonné was not as gentle or polite for the two Lattice Theory pioneers:

> Il semble par contre que les auteurs auraient pu sans inconvénient omettre le chapitre sur les lattices, auxquels toute une école américaine voue une prédilection persistante, malgré le peu d'intérêt que présente cette théorie dans les autres branches des mathématiques. (Dieudonné in Mac Lane et Birkhoff [26, p. XIV] (3)

In the retrospect, how should we interpret such a judgement? Was it a settling of a scores within the mathematical gotha? A struggle for self-promotion between subtopics? As compared with this kind of evaluation and exclusion, we will be more inspired here by the wisdom of another quotation recommending more open-mindedness:

> What's next? (...) In the foreseeable future, discrete mathematics will be an increasingly useful tool in the attempt to understand the world and (...) analysis will therefore play a proportionally smaller role. (...) So, after all that has been said, what's the conclusion? Perhaps in a single word taste. (Halmos [23])

To come back more directly to our present claim, any *lattice* is a *partial order* equipped with two algebraic *operations*: *join* (*upper bound*) and *meet*(*lower bound*), which bear its semantical interpretations. Even in everyday life it is, for example, common to scan through *power sets* ordered by *inclusion*, of which the two operations are *union* and *intersection*, and which are the regular nightly tools of the gambler, experimenter, probabilist (lattice of *events*), and the logician (lattices of *propositions*), or the classifier (*calculus of classes*), since they, often implicitly, structure human activities. (i.e. *Boolean lattices*), other lattices have their own semantics and operations with their own specific structure and properties. They can be as "natural" yet, while being more "arbitrary" and less "regular". Thus, taking *Data Analysis* in a broad sense, the claim can be rephrased as: "*Lattices are natural and basic structures for describing data, which can bear both thought, and calculus*". This argument will be developed throughout the text by following a semi-chronological/thematic approach.

Instead of entering into technicalities, for which the reader is referred to the contributions [55–66], it will be more informative and hopefully more interesting to question the underlying methodological options. Hence, throughout the text, these personal contributions will be matched and put in comparison with some quotations which inspired them, from diverse authors either in Mathematics or in the Philosophy of Science.

2. Lattices on experimental designs

The first non-premeditated encountering between lattices and observational structures dates back to the constitution of Lattice Theory, via Logic and Geometry:

> Il semble à première vue que l'objet du calcul des propositions et celui de la géométrie projective soient différents, et pourtant la structure logique de ces deux disciplines présente (...) bien des analogies. La raison de cette analogie n'est apparue que dans des recherches toutes récentes dans la domaine de ce qu'on appelle l'algèbre abstraite. (Lautman [49, p. 249]) (4)

The original motivation came in 1936 in a pioneering paper by G. Birkhoff and J. von Neumann [5], who were attempting to formalize *The Logic of Quantum Mechanics*:

> The object of the present paper is to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic. Our main conclusion, based on admittedly heuristic arguments, is that one can reasonably expect to find a calculus of propositions which is formally indistinguishable from the calculus of linear subspaces with respect to set products, linear sums, and orthogonal complements – and resembles the usual calculus of propositions with respect to and, or, and not. (Birkhoff and Von Neumann [5, p. 823])

The point which corresponds to our present purposes is that the same family of *modular lattices* happens to underlay the construction of experimental designs, and this is not because of some constraint of the observational space, as in Quantum Physics, but from the a priori requirement to analyse the observations in good conditions of statistical decomposition (uniqueness, independence of the statistic definitions, see [25]. Fig. 1 summarizes the content of [56], which may be stated as follows: the most general experimental designs for which the standard *ANOVA* decomposition is complete and orthogonal belongs to the class of *permuting sublattices of partitions* (on the

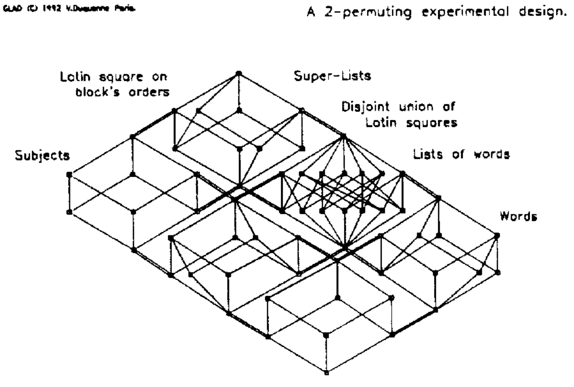
What can lattices do for experimental designs?
Math Social Sciences 11 (1986) 243-281 (MR:87j62140)

Let $x^I:=((x^i)i\in I)\in R^I$, $n_I:=((n_i)i\in I)\in R^{+I}$, $\Phi:=\{A,B,\ldots\}$ be **elementary factors**; let the **experimental design** be $D:=\mathrm{Im}(I \rightarrow \Pi\{F/F\in\Phi\})$, and let $(P_I,\leq,\wedge,\vee,\perp,I,1)$ and $(Q^\perp,\leq,\wedge,\vee,\perp,0,1)$ denote **partitions** on I and **orthogonal projectors** in the Euclidean space R^I equipped with: $<x|y>:\Sigma n_i x^i y^i$. $A\in F$ is evaluated with the **average projector** $mA(x^i):=((x^a)^i)i\in I$ and the **comparison** $cA:=mA-mI\in Q^\perp$.

First, $c:P_I \rightarrow Q^\perp$, $A \mapsto cA$ establishes a Galois connection $P_I \rightarrow [0,cI]$ which makes precise the links **set theoretical description/ANOVA**.

Second, $A,B\in P_I$ are said to be **locally orthogonal** (LO) iff $n_{a\wedge b}=n_a n_b/n_u$ (all $u\in A\vee B$, all $a,b\in u$): this leads to define the **relative interaction** for which $cA\bullet cB:=cA\wedge B-(cA\vee cB)=\oplus_{u\in A\vee B}cA/u.cB/u$.

Third, any LO-subset $X\subseteq P_I$ induces by c a **commutative Boolean subalgebra** $Q(X)\subseteq Q^\perp$, whose atoms give a **canonical ⊕-decomposition**. The set of elementary factors Φ generates a **LO-design** iff the sublattice $P(\Phi)$ is LO; by **Möbius inversion the canonical** ⊖-decomposition is calculated and labelled, extending classic ANOVA **formulas**.



Gluing decomposition in atomic blocks

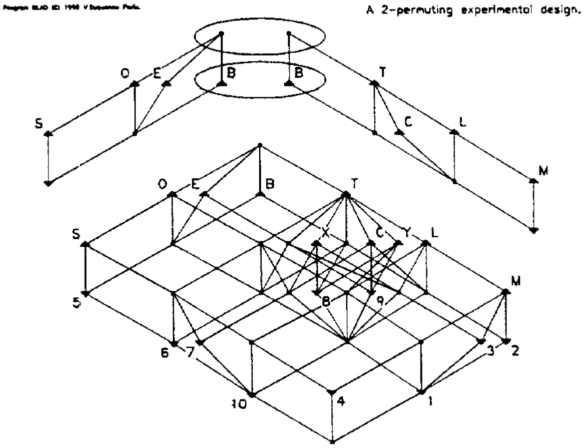


Fig. 1. Permuting designs

data set) which moreover satisfy a condition of *local orthogonality* (two partitions are *permuting* if locally crossed in disjoint blocks, and *locally orthogonal* if the usual Fisher's condition on cardinalities is satisfied in each block). Here we come across the permuting partition lattices, a well-known algebraic structure since the early work by Dubreil and Dubreil-Jacotin [13] to formalize decomposition theorems of algebras, showing that they are modular, which was then extended in a celebrated paper by Ore [28], up to, more currently Jónsson [24], and the characterisation of distributive permuting lattices by Draskovicová [11] (see also [55, 58]).

Fig. 1 displays the modular lattice generated by such a design, encountered in Psychology (see [56, 61]), which can be seen (bottom drawing) as the amalgamation of two sub-structures which correspond to two independent *experimental constraints* and are constructed – in statistical terms – out of "*Latin squares with repetitions*". It is very clear that no experimenter can master this kind of construction in terms of "*n-ary* *tables*" as it can be done for trivial examples: lattices and theorems are required for understanding the design *structure*, process of *generation*, and *analysis*. The top drawing displays the *ungluing decomposition* of the lattice in maximal *atomistic blocks*, thus illustrating a classic theorem by Birkhoff (a kind of decomposition that has been generalized to other lattices by Herrmann, Chajda, Wille and others, by using some *tolerance relations*). In particular, this decomposition points out the location of the *irreducible intervals* which come from the amalgamation of simpler ones (Latin squares).

Then, by using *Möbius inversion* (see [30]) and the characterization of the Möbius functions of modular lattices (see [8]), the statistical terms (*sum of squares, degrees of freedom...*) can be *calculated by recursion* first, and second expressed by a *language of formula* which makes them understandable locally, without requiring the global structure to decipher their meaning. In the general case where several Latin squares are *amalgamated* in an experiment – as in Fig. 1 –, these lattices can become extremely complex. Otherwise, in the *distributive* case for example, structural properties – namely Birkhoff's theorem matching distributive lattices and partial orders, and nice *hereditary properties* of permutability (see [15, 55]) – make them easily generated, and manageable in good conditions. Hence, the corner stone is the presence of Latin squares and other confusion structures in a design, which R.A. Fisher justified in his book devoted to *The Designs of Experiments* as follows:

> If large quantities of material are needed, or large numbers of laboratory animals, these will almost invariably be more heterogeneous than smaller lots could be made to be. In like manner, extensive compilations of statistical material often show evidence of such heterogeneity among the several parts which have been assembled, and are seriously injured in value if this heterogeneity is overlooked in making the compilation. (...) In such cases we may usefully adopt the artifice known as confounding. (Fisher [17, p. 107])

The constraints generating confoundings lead to an abstract complexity that is managed by permuting lattices, which in turn secure simple and recursive analysis:

> As Tjur [34] and Duquenne [56] have shown, variance models based on [locally-orthogonal designs] have a straightforward analysis of variance, which parallel much of the present paper. (...) The coefficients can be easily calculated recursively, using the semilattice, whereas there is no such simple method for calculating those coefficients for a general association scheme. (Bailey in Speed [32, p. 914])

In a paper parallel to [55, 56], ... and following a more statistical line, T. Tjur expressed precisely how – for this class of locally orthogonal designs – linear aspects and matrix calculus are secondary and displaced by more structural problems:

> This paper deals with analysis of variance (ANOVA) models in experimental designs where all factors (treatment factors as well as blocking) are orthogonal. (...) Mathematically, this is a rather exclusive class of experimental designs.

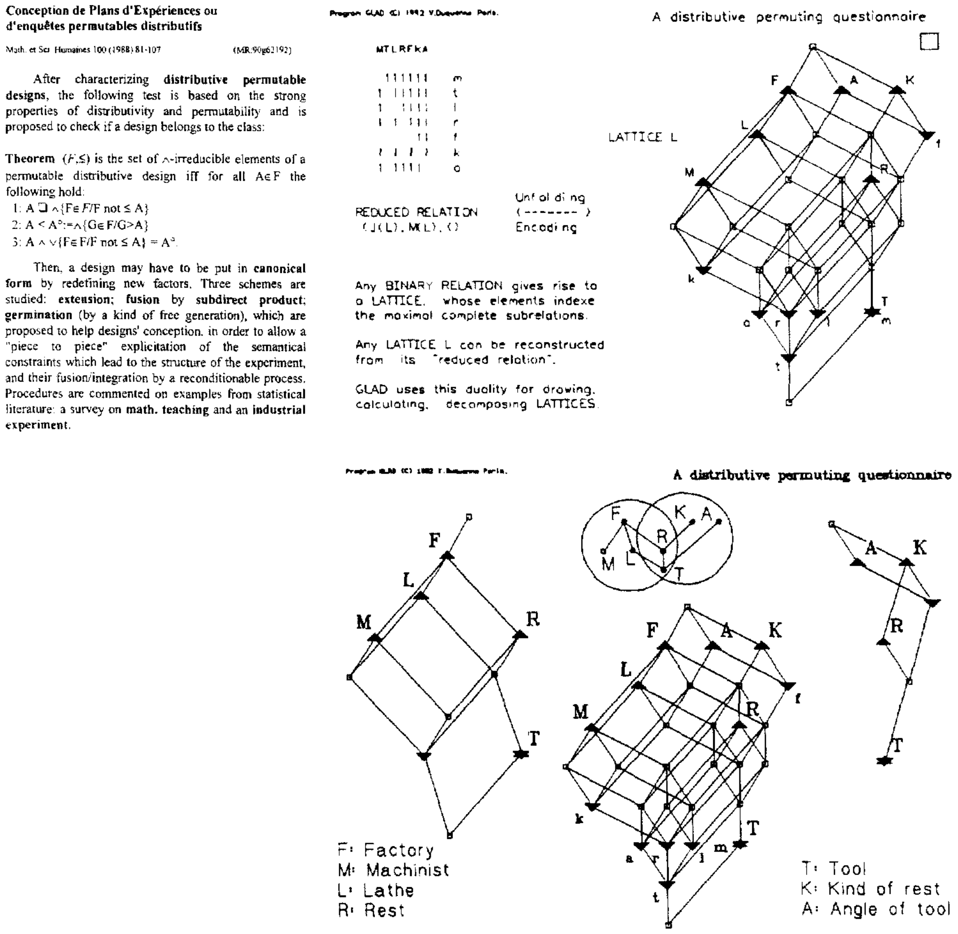
> Statistically, however, it is very important, and more or less standardized methods for the handling of analysis of variance models in such designs are given by most books on experimental design. (...) The main tool of the modern approach to analysis of variance is matrix calculus based on the concepts from Euclidean geometry (orthogonal projections, etc.). However, in the case of orthogonal designs it is generally recognized that the matrix calculations involved are purely formal, in the sense that the interpretation of the symbols as matrices plays a secondary role. The final results (e.g. formulae for sums of squares of deviation) are not stated in terms of matrices anyway, and the intermediate matrix manipulations can be more or less replaced by similar operations on other algebraic objects, like symbolic expressions ... (Tjur [34, p. 33])

After the announcement of the pending achievement of a general classification of permuting designs [56], an event disrupted the project when Jónsson's [24] long-standing conjecture on the assimilation of permuting to Arguesian lattices was invalidated:

> A lattice L is linear if it is representable by commuting [permutable] equivalence relations. Jónsson showed that any such lattice is Arguesian. Numerous equivalent forms of the Arguesian law are now known; it is a strong condition with important applications in coordinatization theory. Nevertheless, the question raised by Jónsson, whether every Arguesian lattice is linear, has remained open until now. (...) The results of Section 3 imply that no finite set of identities (...) can completely characterize linearity; in particular, the Arguesian law is insufficient. (Haiman [22, pp. 121–123])

If the invalidation of Jónsson's conjecture was in some sense disappointing for the community working on lattice decomposition, the consequences in terms of designs are not so terrible: it remains possible to focus on sub-classes which are manageable (see [58, 61]), and to graft on these designs a few exotic sub-structures that do not make them explode. In particular, in the distributive case, as illustrated in Fig. 2, a Theorem gives an efficient test for checking whether a design is permuting, and several generation processes by *free construction* and *amalgamation* are provided as good candidates to formalize naturally how an experimenter can conceive a design, by bringing together local constraints. For instance, the bottom drawing of Fig. 2 illustrates how this example of the statistical literature was actually generated by *subdirect product* of two components, describing respectively the *tools* T, and their relation with the *machinists* M *within* the *factories* F.

To confront the dilemna that locally orthogonal designs constitute a nice class, statistically "straightforward" (Bailey in [32]) but somehow algebraically untractable in the general permuting class without other restriction, a statistician feeling annoyed by the absence of more solid mathematical protections for building designs, could still follow one of the following pieces of advice: the first one is given by two experts in



Conception de Plans d'Expériences ou d'enquêtes permutables distributifs

Math. et Sci. Humaines 100 (1988) 81-107 (MR:90g62192)

After characterizing **distributive permutable designs**, the following test is based on the strong properties of distributivity and permutability and is proposed to check if a design belongs to the class:

Theorem $(F,\leq)$ is the set of $\wedge$-irreducible elements of a permutable distributive design iff for all $A \in F$ the following hold:

1: $A \sqsupset \wedge\{F \in F / F \text{ not} \leq A\}$

2: $A < A^{\circ} := \wedge\{G \in F / G > A\}$

3: $A \wedge \vee\{F \in F / F \text{ not} \leq A\} = A^{\circ}$.

Then, a design may have to be put in **canonical form** by redefining new factors. Three schemes are studied: **extension**; **fusion** by **subdirect product**; **germination** (by a kind of free generation), which are proposed to help designs' conception, in order to allow a "piece to piece" explicitation of the semantical constraints which lead to the structure of the experiment, and their fusion/integration by a reconditionable process. Procedures are commented on examples from statistical literature: a survey on **math. teaching** and an **industrial experiment**.

Program GLAD (C) 1992 V.Duquenne Paris.

```
 MTLRFkA

  111111    m
 1 11111    t
 1   1111   l
 1 1 111    r
      11    f
 1 1 1 1    k
 1 1111     o
```

REDUCED RELATION (J(L), M(L), ()

Unfol di ng
(--------)
Encodi ng

Any BINARY RELATION gives rise to a LATTICE, whose elements indexe the maximal complete subrelations.

Any LATTICE L can be reconstructed from its "reduced relation".

GLAD uses this duality for drawing, calculating, decomposing LATTICES.

A distributive permuting questionnaire

Program GLAD (C) 1992 V.Duquenne Paris.

A distributive permuting questionnaire

F: Factory
M: Machinist
L: Lathe
R: Rest

T: Tool
K: Kind of rest
A: Angle of tool

Fig. 2. Distributive permuting designs

experimental designs, the second recommendation, more ambitious, expresses a need to proceed with the formalization of new classes:

> Participation in the initial stages of experiments in different areas of research leads to a strong conviction that too little time and effort is put into the planning of experiment. The statistician who expects that his contribution to the planning will involve some technical matter in statistical theory finds repeatedly that he makes a much more valuable contribution simply by getting the investigator to explain clearly why he is doing the experiment, to justify the experimental treatments whose effects he proposes to compare, and to defend his claim that the completed experiment will enable its objectives to be realized. (Cochran and Cox [6, p. 10])
> Further investigation in this area seems needed. The names of the designs do not matter: what is required is to identify the right class of design. By right I mean: (i) capable of being meaningful in relation to real treatments, (ii) capable of being

simply described to a computer, (iii) permitting enough theory to be developed about the whole class that both design and analysis can be simplified. Although the literature on design is so vast, very little of it satisfies all three criteria. (Bailey in Tjur [34, p. 73])

3. GLAD: general lattice analysis and design

To be in a better position to construct, decompose, and draw ... finite lattices coming from experimental designs, from mathematical problems, as well as from the analysis of binary relations generated by observational data, the computer program *GLAD* has been developed, since 1983. The manual defining its commands has been described at length in [66], to which the interested reader is referred, and the contributions [59–66] make use of its possibilities. Fig. 3 lists some global descriptions of the commands and illustrates two typical outputs. To comment upon them more precisely, let us shortly recall the two fundamental dualities on which the program is built:

Lattices as conceptual models: first, lattices can be useful for unfolding *binary relations* (*dichotomic data*, represented by *01-matrices*) into the set of their maximal blocks (filled with 1 in the matrix), and in so doing give a frame for formalizing the *duality extension/intension* of concepts. Historically, as said before, the ability to represent any kind of duality through lattices had been pointed out and stressed by Lautman [49] in the context of Philosophy of Mathematics. From a technical viewpoint, the underlying mathematical structures have been well known since Birkhoff [3] and the recognition of their potential usefulness in Data Analysis was stressed by Barbut and Monjardet [1] which has been followed by technical developments in [38] and in the contributions [57, 66]. A lattice is a *partial order* in which every pair of element (x, y) has a *meet* denoted by $x \wedge y$, and dually a *join* $x \vee y$. Remarkable elements can be distinguished, among which the *meet*-and *join-irreducible* elements, which respectively have a unique *upper cover* and a unique *lower cover*, and generate the lattice downwards, and upwards. For a lattice L, let $J(L)$ and $M(L)$ be its sets of join and meet-irreducible elements, and for an element x of L let denote by J_x the subset of join-irreducible elements below x, and let M_x, dually, be the set of meet-irreducibles above x. A fundamental if obvious remark is that:

the higher is x, the larger is J_x, while M_x is smaller,

a mechanism that can directly handle the duality extensions/intensions, as it has been described from Aristotle up to the more precise formulation in the *Port Royal Logic* by Arnault and Nicole, in the 17th century, which already stressed the fact that the more a concept's intension grows, the smaller its extension must be, and conversely.

From a technical viewpoint, the structure of a lattice L is obviously encoded into the order relation on irreducibles by the bijection $x| \rightarrow (J_x, M_x)$ (all $x \in L$), since x is less or equal to y iff J_x is a subset of J_y (iff M_x is a superset of M_y); $J_{x\wedge y} = J_x \cap J_y$ and $M_{x\vee y} = M_x \cap M_y$ hold in any (arbitrary) lattice, while $J_{x\vee y}$ is a superset of $J_x \cup J_y$ and $M_{x\wedge y}$ is a superset of $M_x \cup M_y$ generally (the equalities hold for *distributive*

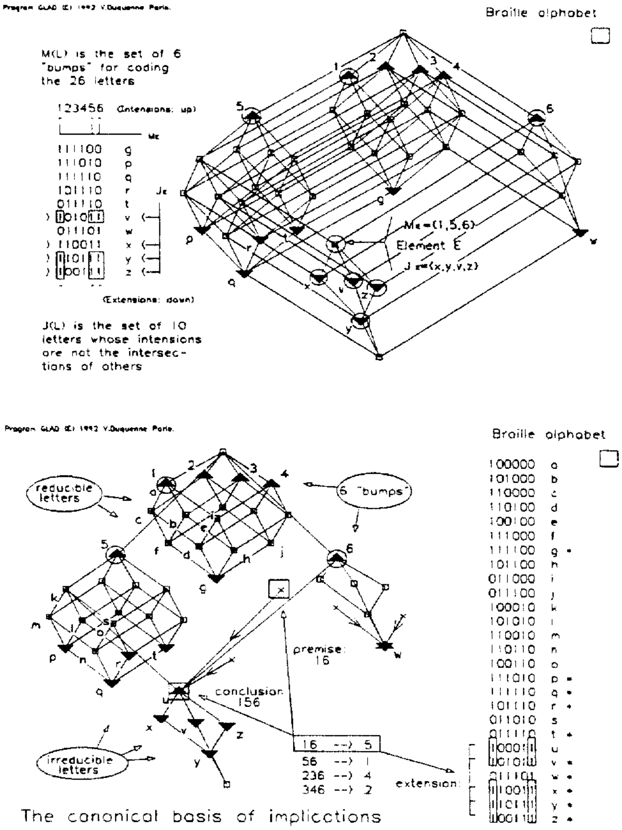
GLAD (General Lattice Analysis and Design)
A Fortran program for a glad user (1983-96)

ORDAL'96 (I. Rival ed.), 1996, Ottawa (see www.csi.uottawa.ca/ordal)

GLAD is a program for editing, drawing, modifying, decomposing, ..., approximating (finite) lattices, which may come either from abstract Mathematics, of from applied Statistics like Analysis of Variance, or from the need to analyze 01-data, by exploiting the classic correspondence **lattices/binary relations**.

One of the main feature of **GLAD**, through the small utility program VgaSHOW, is to provide the possibility to make *slide presentations* on Vga screens. Dozens of images can be compressed on a diskette and easily extracted in real time without the user intervention, for organizing *lectures*, *exhibitions*, *conferences*.

The main commands of **GLAD**:
- **zoom**: zooming, moving, centering the lattice.
- **core**: meet-core, meet-irreducible elements.
- **box**: box decomposition to simplify a drawing.
- **gluing**: blocks of a **tolerance** relation.
- **slat**: calculates, extracts sublattices.
- **histo**: represents wheighted lattices.
- **approx**: approximates weighted lattice.



M(L) is the set of 6 "bumps" for coding the 26 letters

```
123456  (Intensions: up)
               Mε
111100   g
111010   p
111110   q
101110   r    Jε
011110   t
) 101011 v  (—
011101   w
) 110011 x  (—
) 110111 y  (—
) 100111 z  (—
     (Extensions: down)
```

J(L) is the set of 10 letters whose intensions are not the intersections of others

```
16   --> 5
56   --> 1
236  --> 4
346  --> 2
```

```
100000  a
101000  b
110000  c
110100  d
100100  e
111000  f
111100  g *
101100  h
011000  i
011100  j
100010  k
101010  l
110010  m
110110  n
100110  o
111010  p *
111110  q *
101110  r *
011010  s
011110  t *
100011  u
101011  v *
011101  w *
110011  x *
110111  y *
100111  z *
```

Fig. 3. General lattice analysis and design

lattices, which are isomorphic to set of subsets closed for union and intersection, and embeddable in *Boolean* lattices). These basic notions make explicit the *dual* interplay between binary relations and lattices (for more details, see [3, p. 124]), and explain how what can be said about "concepts" can be directly stated in terms of Lattice Theory. Hence, the upper drawing in Fig. 3 illustrates the mapping $x| \longrightarrow (J_x, M_x)$ (all $x \in L$): the lattice element denoted by E, which is pointed to by a cycle, is dominating the join-irreducible elements $\{v,x,y,z\}$, and is dominated by the meet-irreducibles $\{1,5,6\}$. The corresponding sub-table is indicated at the left-hand side, as a caption that is pasted into the drawing.

Coming back to our present topic (and conversely to the lattice encoding), let R be a "concrete" relation defined as a subset of $O{\times}A$ that represents the description of a set of objects O, by a set of attributes A:oRa is read "the object o *receives* the attribute a", as well as, conversely "the attribute a *applies* to the object o". For a subset of attributes $B \subseteq A$, let $B\downarrow = \{o \in O | oRb \,\mathrm{all}\, b \in B\}$ define its "*extension*", which is the sets of all object to which B applies, and $B\downarrow\uparrow = \{a \in A | oRa \,\mathrm{all}\, o \in B\downarrow\}$ its "*intension*", dually. The structure of $R \subseteq O{\times}A$ is unfolded into the *Galois lattice* $L(R) = \{(B\downarrow, B\downarrow\uparrow)|$ all subsets B of $A\}$. It is sometimes useful to consider these *maximal blocks* $(B\downarrow, B\downarrow\uparrow)$ of the relation R as concepts, the *extension* and *intension* of which consist of $B\downarrow$ and $B\downarrow\uparrow$. Due to the previous remarks, the union of intensions is not always an intension, and the same applies to extensions (the contrary would force all lattices to be distributive). More formally, the meet and join operations of the Galois lattice $L(R)$ are defined as

$$(B\downarrow, B\downarrow\uparrow) \wedge (C\downarrow, C\downarrow\uparrow) = ((B\downarrow \cap C\downarrow), (B\downarrow \cap C\downarrow)\uparrow)$$

and

$$(B\downarrow, B\downarrow\uparrow) \vee (C\downarrow, C\downarrow\uparrow) = ((B\downarrow\uparrow \cap C\downarrow\uparrow)\downarrow, (B\downarrow\uparrow \cap C\downarrow\uparrow)).$$

In words, the *meet* operation (down, for defining lower bounds) corresponds to the *intersection of extensions*, while the *join* operation (going up in the lattice) corresponds to the *intersection of intensions*, dually. Calculating meets/joins can be done in Fig. 3.

Lattices as implication models: For defining this second duality, let $B \subseteq A$ be a subset of attributes, and let us also denote by B the *conjunction* of the properties "*having the attributes b that B* comprises". Implications between conjunctions of attributes are defined by: *B implies C*, denoted by $B \rightarrow C$, if $B\downarrow$ is a subset of $C\downarrow$, i.e. if all the objects having the conjunction of properties of B do have those of C as well. This is consistent with the common usage: conjunctions are just replaced by subsets of A for brevity. Now, some implications do not depend on the relation R and are simply the consequences of the *propositional calculus*: for instance, if B is a superset of C, then $B \rightarrow C$ holds, since $B\downarrow$ must be a subset of $C\downarrow$. Such implications can be read in the lattice, and express either ordering between attributes, or synonymies between complex conjunctions of attributes that make the observed Galois lattice smaller, so that there is a *duality implications/lattices*: the more L(R) is far from being of the form $2**\#A$, the more there exist implications which reflect *synonymies* between conjunctions of attributes (i.e. which share the same extensions).

It has been shown that all the implications holding in R can be inferred from a *canonical basis* (and that all basis have the same cardinality, see [21, 57], which is *non redundant* for the following set of *inference rules* (which defines a restricted propositional calculus, without *negation* and *contraposition*):

$B \rightarrow D$ is inferred from $B \rightarrow C \,\&\, C \rightarrow D$;

$B \rightarrow C$ is inferred from $C \subseteq B$;

$B \cup X \rightarrow C \cup X$ (all $X \subseteq A$) are inferred from $B \rightarrow C$.

This implications/lattices duality is of a fundamental nature for encoding interpretable information equivalent to the original data (up to the objects' labels), for representing pragmatic implications between *conjunctions of attributes* (or *conjunctions of objects*).

The duality implications/lattices is illustrated in the bottom drawing of Fig. 3. For drawing clarity, the lines parallel to the ones linking 5 and 6 to the lattice top have been erased (which resembles Wille's [38] *boxing* procedure). For instance, the implication 16→5 can be checked in the table (all objects receiving 1 and 6 also receive 5) as well as in the lattice (the meet of 1 and 6 is below 5, and therefore captures it as a "new" meet-irreducible in the corresponding "intension").

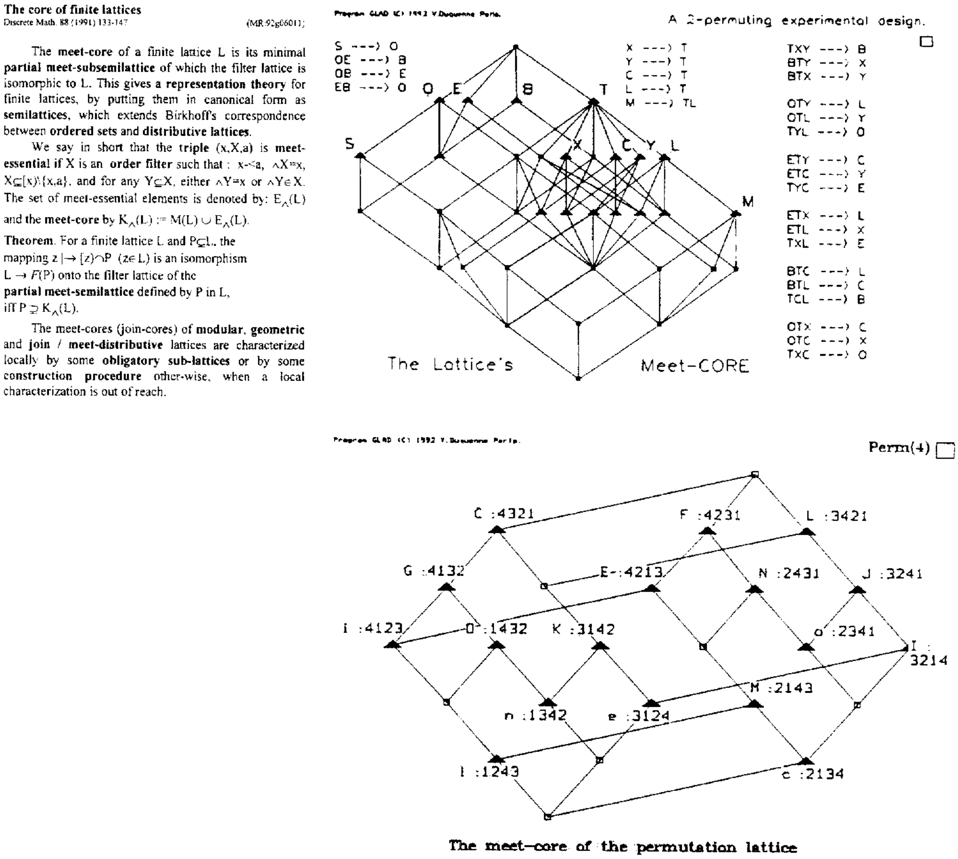
4. Lattices cores and lattices of permutations

To report now on a more algebraic direction of this work, after having rubbed shoulders with many a lattice coming from experimental designs, a natural question which comes to the surface is: *how can we define a lattice minimally?* Of course, we already had the *incident relation between* join and meet-*irreducible elements*, and the *Galois lattice* construction. On the other hand, we had defined (see [21, 57]) the *canonical basis* of implications, which is a kind of *canonical form* for any *closure operators*, and can therefore be used to represent finite meet-semi-lattices. But these representations are "external" to the lattice structure (see also [10, 35, 36]); in the same direction of lattice representations, see also [9]. Hence the question could be refined as: *how can we define a lattice in a minimal way "internally", in term of its substructures*? For this, we needed the formalism of *partial semi-lattices* (developed in [19, 29, 37]).

The concept of *meet-core* of a (finite) lattice L, which is the minimal sub-structure out of which the lattice L can be re-generated by the meet operation, was introduced in [57, 59], and leads to a generalization of G. Birkhoff's Theorem exhibiting any *distributive lattice* D as dually isomorphic to the *filter lattice* of its order of meet-irreducible elements $(M(D), \leqslant)$. The meet-core consists in the union of the sets of *meet-irreducible* elements and of *meet-essential* elements which satisfy the technical condition recalled in Fig. 4. Unfortunately, for an arbitrary finite lattice, there will not be any *local criterion* for checking the property that an element x is meet-essential or not: you need the whole filter $[x,1]$. On the other hand, the meet-core is a subset of the *scaffolding* (see [37], and in general is smaller than the scaffolding in the case of a *subdirectly irreducible lattice*, since the scaffolding is then always equal to the lattice.

In the case of modular lattices, such a local criterion exists, and the meet-core is equivalent to the well known *base of lines* (see [2, 35, 36] namely, an element is meet-essential if and only if the interval generated by its upper-covers is an M_n $(n \geqslant 3)$. The top drawing of Fig. 4 thus represents the meet-core of the abstract lattice generated by the experimental design of Fig. 1. It is quite meaningful when we reinterpret it in the lattice of partitions (on the data set): we get a kind of "canonical form" of our experimental design by defining the order of meet-irreducible partitions and the Latin squares corresponding to the meet-essential partitions. For other classes of lattices, local conditions are quite tricky: the interested reader is referred to [57, 59].

Permutations on a set play a prominent role in many fields of Mathematics, as well as more applied topics like Statistics, Data Analysis, Experimental Designs and Genetics, or kinship systems in Anthropology. The first property which is emphasized in every



The core of finite lattices
Discrete Math. 88 (1991) 133-147 (MR:92g06011)

The **meet-core** of a finite lattice L is its minimal **partial meet-subsemilattice** of which the filter lattice is isomorphic to L. This gives **a representation theory** for finite lattices, by putting them in canonical form as **semilattices**, which extends Birkhoff's correspondence between **ordered sets and distributive lattices**.

We say in short that the **triple** (x,X,a) is **meet-essential** if X is an **order filter** such that : $x \prec a$, $\wedge X = x$, $X \subseteq [x) \setminus \{x, a\}$, and for any $Y \subseteq X$, either $\wedge Y = x$ or $\wedge Y \in X$. The set of meet-essential elements is denoted by: $E_\wedge(L)$ and the **meet-core** by $K_\wedge(L) := M(L) \cup E_\wedge(L)$.

Theorem. For a finite lattice L and $P \subseteq L$, the
mapping $z \mapsto [z) \cap P$ $(z \in L)$ is an isomorphism
$L \to \mathcal{F}(P)$ onto the filter lattice of the
partial meet-semilattice defined by P in L,
iff $P \supseteq K_\wedge(L)$.

The meet-cores (join-cores) of **modular**, **geometric** and **join / meet-distributive** lattices are characterized locally by some **obligatory sub-lattices** or by some **construction procedure** other-wise, when a local characterization is out of reach.

The meet-core of the permutation lattice

Fig. 4. The meet-cores of lattices

book, is that starting from any permutation $1 = (123\ldots)$ on a set $n = \{1, 2, \ldots, n\}$, any permutation can be obtained by a series of transpositions reversing neighbors. This gives rise to the symmetric group on an n element set.

It was observed long ago in the context of Social Choice (see [1]) that the *transpositions* also defines the *cover relation* of a lattice, which is denoted by Perm(n). If the group theoretic properties of permutations are well known, some lattice properties of Perm(n) are still not so clear (see [60]). The main results are recalled in Fig. 5, which aims at contributing to fill the gap and to the revival of interest in permutation lattices. The, quite messy top drawing of Fig. 5 is the first drawing which was obtained with *GLAD*, while the regular bottom drawing was obtained after a while, by embedding Perm(5) into a "pseudo product" of chains: $1 \times 2 \times 3 \times 4 \times 5$.

The meet-core of Perm(n) is characterized, and meet-essential elements turn out to be locally characterized: it is shown that a permutation x is meet-essential in Perm(n) if and only if it has two covers a, b such that $[x, a \vee b]$ is isomorphic to Perm(3). As a corollary, a permutation is meet-essential in Perm(n) if and only if it has exactly

On permutation lattices
Math. Social Sciences 27 (1993) 73-89 (MR:95g20004)

The lattice Perm(n) of **permutations** on an n-element set n={1,...,n}, "rooted" at (1...n), is shown to be **meet** and **join-semidistributive**. which implies known results such as the non-existence of M_3-sublattices, and that the complementation defines a congruence relation with $2^{|J(Perm(n))|}$ classes.

The **meet-core** of Perm(n) is shown to be the set of meet-irreducible elements together with all the elements that have two upper-covers that moreover generate a covering sublattice isomorphic to Perm(3); this expresses that the meet operation is completely expressible in terms of reversing **adjacent pairs**.

A **recursive construction** of Perm(n) as a **Galois Lattice** -via a kind of summation process- is given, which has been a key for obtaining clear drawings of Perm(4) and Perm(5) with our new PC/Vga graphic program **GLAD** (General Lattice Analysis & Design).

Last, the (distributive) **congruence lattice** C(Perm(n)) is recursively characterized, through the order of its meet-irreducible elements.

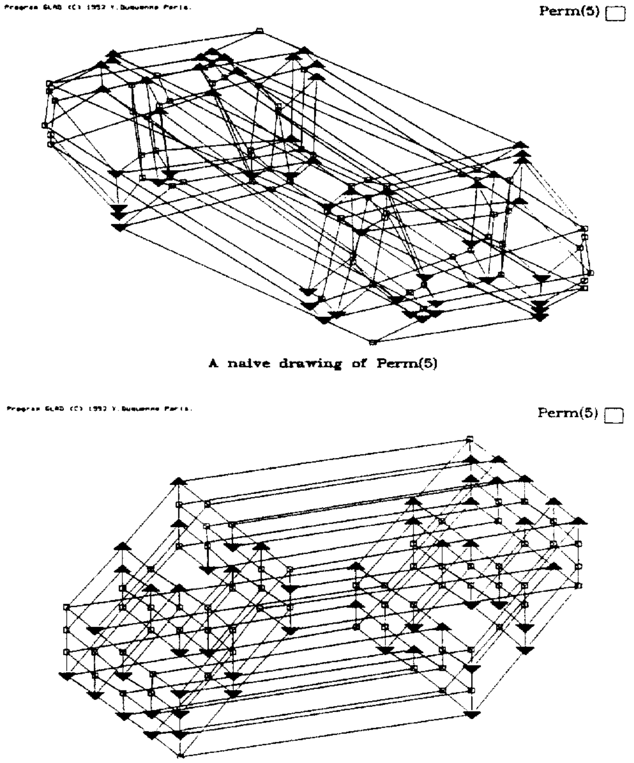


Fig. 5. Two drawings of Perm(5)

two pairs of neighbors that are adjacent and not in alphabetic order. Hence, all the equalities between meet-expressions can be reconstructed in terms of adjacent pairs.

More technical properties of Perm(n) can be explored, of which a lot are the consequence of its *join/meet-semidistributivity*, for which the reader is referred to [60].

5. Representations, implications, abstraction

Let us – in a neutral way – call *Lattice Analysis* of dichotomic data all the procedures and technics which aim at analysing the structure of a given "observed" binary relation, by using the Galois lattices' model, the canonical basis of implications and the properties described above, which are implemented in the program *GLAD*.

As an example of output of Lattice Analysis, Fig. 6 (extracted from [64]) represents the description, by eight global attributes, of around 3000 children which have been excluded from the French school system due to severe psychological or cerebral illnesses. The diseases *I/K* and *C/P* being exclusive by construction (respectively *cerebral/non-cerebral palsy*, *neurosis/psychosis*), it happens that nearly all the potential conjunctions are observed in that population, which is not that surprising considering

Lattice analysis and the representation of handicaps associations

Social Networks 18 (1996) 189-199

A main goal of this paper is to show how **Lattice Analysis** and the recent computer program **GLAD** can help in understanding the associations between **psychological handicaps**, for a population of children that were excluded from school, due to severe illnesses (combinations of: *blindness*, *hearing disorders*, *neuroses*, *cerebral/non cerebral palsy*, *epilepsy*, *cardiac disorders*).

After very few technical recalls, it is shown how lattices can be taken as a **conceptual** as well as an **implicational model** of *multi-handicaps*, thanks to their natural ability to formalize the definition of illnesses in terms of either **extension** or **intension**.

It is then shown how some **sub-structures** of the lattice, **weighted** by the cardinalities of the patient groups, can display the **assessed associations** between *handicaps*, addressing quite directly the original question raised by the *health planner*.

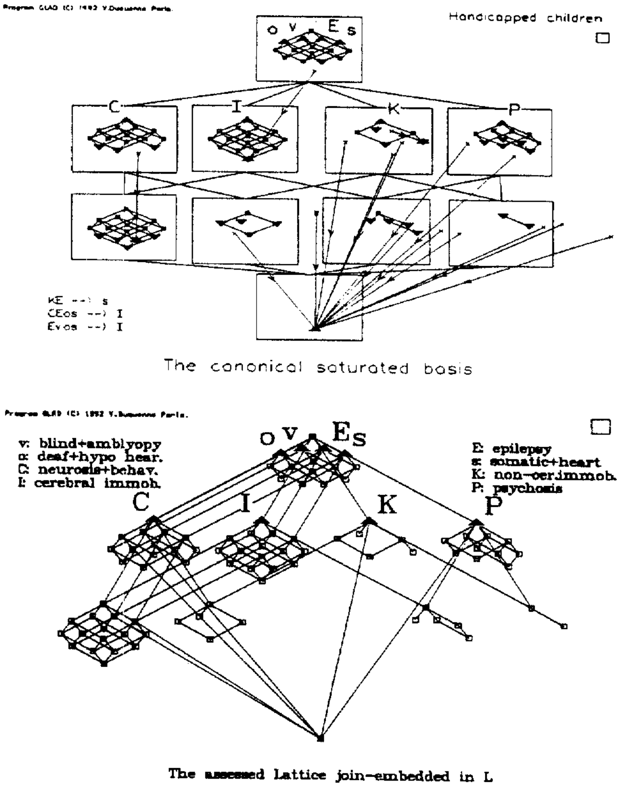


Fig. 6. Handicapped children excluded from school

the size of the population compared to the small number of attributes. The bottom lattice is obtained from the top one by removing the intensions of which the extensions are small, namely less than 1% of the population (this threshold is arbitrary, and could be changed). In this "approximated cut" lattice, the attribute P (psychosis) does not associate with the somatic attributes (o, v, E, s), which is quite coherent with the basic observations in Psychiatry. In the top drawing of Fig. 6, the *canonical basis of implications* is nested into the *boxed* lattice. Only three implications are born by an intension which is a proper subset of attributes. All the other implications point to counterexamples, since no patient has all the attributes. Hence, there are not many implications, in this example, out of those expressing the split between what is observed and what is not. Before commenting on further examples, we will focus for a while on the aims and methodology which are implicitly underlying Lattice Analysis.

A first positive advantage of using Galois lattices to represent and analyse dichotomic data is to *equally treat* intentional and extensional interpretations, which are sent back to back by the underlying *Galois correspondence*. This possibility of using a re-unified language was called for by many precursors of the Philosophy of Science:

> Les deux interprétations logiques. Il se présente même une circonstance particulièrement intéressante: l'Algèbre en question est susceptible, en Logique même, de deux interprétations distinctes, dont le parallélisme est presque parfait, suivant que les lettres représentent des concepts ou des propositions. Sans doute, on peut, avec Boole et Schröder, ramener ces deux interprétations à une seule, en considérant les concepts, d'une part, et les propositions, d'autre part, comme correspondant à des ensembles ou classes: un concept détermine l'ensemble des objets auxquels il s'applique (et qu'on nomme en Logique son extension); une proposition détermine l'ensemble des cas ou des instants du temps où elle est vraie (et qu'on peut, par analogie, appeler aussi son extension); et alors le calcul des concepts et celui des propositions se réduisent à un seul... (Couturat [7, p. 3]) (5)

> Many systems have different names for properties and for the corresponding classes. This is discussed with respect to examples from the system of Principia Mathematica. Analysing these names by the method of extension and intension, we find that a name for the property Human and a different name for the class Human have not only the same extension but also the same intension. Therefore, the duplication of names to which the method of the name-relation leads is superfluous. (Carnap [44, p. 106])

Second, if the Galois machinery so cleverly assigns a symmetric role to the intensional/extensional components of "concepts", in order to extend our knowledge it is sometimes practical or necessary to take a much more oriented *disymmetric view*:

> The notion of the extension of a concept, as the class of those objects to which the concept refers, is most often taken to be more transparent than that of intension, however sharply the medieval logicians conceived the distinction. Our proof-theoretic demystification of the intension of a concept, in terms of the class of those concepts which are logically derivable from the given concept, supplies a second definite pole, or axis, relative to which meaning may be conceived; linguistic axis for meaning which proves to be, in a surprisingly precise sense, dual to that of reference. The flexibility of this dualistic view of meaning as comprised of intensional and extensional components articulated such linguistic and referential (in technical logical terminology, syntactical and semantical) poles, holds out sound promise as a tool for the elucidation of meaning in general, and, by laying sufficient stress on deduction, suggests an appropriate explanation of mathematical objectivity. (Castongay [45, p. 2])

However, unanimity was not reached on the questions of abstraction, specially in the middle of this century when several controversies arose with other schools which even doubted the possibility of clarifying the intensional meaning of "concepts". In particular:

> The purpose of this paper is to defend the thesis that the analysis of intension for a natural language is a scientific procedure, methodologically just as sound as the analysis of extension. To many linguists and philosophers this thesis will

> appear as a truism. However, some contemporary philosophers, especially Quine and White believe that the pragmatical intension concepts are foggy, mysterious, and not really understandable, and that so far no explications for them have been given. (...) They emphasize that their objection against the intension concepts is based on a point of principle and not on the generally recognized facts of the technical difficulty of linguistic investigations, the inductive uncertainty, and the vagueness of the words of ordinary language. (Carnap [44, p. 236])

By matching the intensions and extensions, Galois lattices provide a representation which avoids this doubling language, and therefore is in itself a reduction. To reduce a problem complexity is already an accomplishment, but more than providing a mere linguistic economy, avoiding this doubling aims at establishing the right conditions for mastering the *dynamic of argumentation*, by making use of either examples, or rules.

> L'esprit scientifique peut se fourvoyer en suivant deux tendances contraires: l'attrait du singulier et l'attrait de l'universel. Au niveau de la conceptualisation, nous définirons ces deux tendances comme caractéristiques d'une connaissance en compréhension et d'une connaissance en extension. (...) Il faudrait créer ici un mot nouveau, entre compréhension et extension, pour désigner cette activité de la pensée empirique inventive. Il faudrait que ce mot pût recevoir une acceptation dynamique particulière. En effet, d'après nous, la richesse d'un concept scientifique se mesure à sa puissance de déformation. Cette richesse ne peut s'attacher à un phénomène isolé qui serait reconnu de plus en plus riche en caractères, de plus en plus riche en compréhension. Cette richesse ne peut s'attacher davantage à une collection qui réunirait les phénomènes les plus hétéroclites, qui s'étendrait, d'une manière contingente, à des cas nouveaux. La nuance intermédiaire sera réalisée si l'enrichissement en extension devient nécessaire, aussi coordonné que la richesse en compréhension. Pour englober des preuves expérimentales nouvelles, il faudra alors déformer les concepts primitifs, étudier les conditions d'application de ces concepts et surtout incorporer les conditions d'application d'un concept dans le sens même du concept. (Bachelard [43, p. 60]) (6)

Thus, seeking to formalize abstraction supposes a clear recognition of the intension/extension duality giving priority to intensions, which is somehow a revival of original attitudes in the Philosophy of Science. For instance, the experts seem to agree that the fundamental turn in conceptualizing the duality intension/extension had been drawn by the *Logique de Port Royal*, at the end of the 17th century:

> En général la discussion sur les propositions simples et les propositions complexes, tout au long de la Grammaire et de la Logique de Port-Royal, suggère un concept d'idée ainsi entendu ; en effet, les propositions sont décrites comme une combinaison d'idées, et on dit des idées complexes qu'elles ont pour base les propositions constituantes sous-jacentes. En ce sens, le mot idée est un terme théorique appartenant à la théorie des procès mentaux; la compréhension (c'est à dire l'intention ou la signification) d'un idée est la notion fondamentale pour

> l'interprétation sémantique, et dans la mesure où l'on considère que la structure profonde du langage est un reflet direct des procès mentaux, c'est aussi la notion fondamentale pour l'analyse de la pensée (Chomsky [46, p. 66]) (7)

The same evaluation is also found in a book on *Meaning and existence in Mathematics*:

> The Port Royal treatment of intension is also significantly original. The interconceptual relation which determines intension is enlarged far beyond the simple relation of genus to species, to encompass a more sophisticated logical organization of universals. The inclusion, in the intension of the idea of a triangle, of the fact that the sum of the angles of a triangle equals two right angles, involves the entire logical apparatus of Euclidean geometry in the determination of intension. This opening of the traditional doctrine to include more complex logical relations than mere predication in the determination of intensions, suggests the following evaluation of the Port Royal doctrine. (Castongay [45, p. 10])

With such deep roots in the Philosophy of Science, a third natural if more technical question is: how does Lattice Analysis *compare with traditional procedures* in Statistic? All statistical methods commonly used in Behavioral and Social Sciences do not fall into linear methods such as *Analysis of Variance*, *Factorial Analysis* and the like. From the first development of Statistics, the distinction was even made between *evaluation procedures* on numerical variables, and on the other hand *counting procedures* based on qualitative variables, often discrete, which are mathematically simpler:

> The quantitative character may arise in two different ways. In the first place, the observer may note only the presence or absence of some attribute in a series of objects or individuals, and count how many do or do not possess it. (...) The quantitative character, in such cases, arises solely in the counting. In the second place, the observer may note or measure the actual magnitude of some variable character for each of the objects or individuals observed. (...) The observations in these cases are quantitative ab initio. (...) But the methods and principles developed for the case in which the observer only notes the presence or absence of attributes are the simplest and most fundamental, and are best considered first. (Yule and Kendall [39, p. 11])

Since they are more basic – if not more fundamental – , the discrete methods of which the dichotomic data are defined in terms of presence/absence of attributes offer the advantage of *avoiding the difficult questions of axiomatisation*, which are encountered with quantitatification, and the development of *Measurement Theory*. On the other hand, simpler mathematics do not mean simpler methods or even implistic conclusions:

> When it comes to detecting order in relatively well-structured domains, however, like kinship and social organisation, beliefs systems, economic exchange and material possessions, discrete methods like graph theory, Boolean algebra and lattice

> analysis are very apt because they detect subtle structure and remain close to the raw data. (Schweizer [54, p. 250])

The fourth question concerns the old philosophical *problem of reduction*: whenever one considers a set of – concrete or abstract – objects, described in terms of presence/absence of attributes, an immediate question which arises – all the more that discrete methods often explode combinatorially – is the question of reducing this description by removing redundency as much as possible, and this both at the basic level of description of events, and at a more global level of construction of scientific theories:

> There are as a rule a number of ways in which the words used in a science can be defined in terms of a few among them. These few may have ostensive definitions, or may have nominal definitions in terms of words not belonging to the science in question (...). Such a set of initial words I call "minimum vocabulary" for the science in question, provided that (a) every other word used in the science has a nominal definition in terms of these words, and (b) no one of these initial words has a nominal definition in terms of the other initial words. Everything said in a science can be said by means of the words in a minimum vocabulary. For whenever a word occurs which has a nominal definition, we can substitute the defining phrase; if this contains words with a nominal definition, we can again substitute the defining phrase, and so on, until none of the remaining words have nominal definitions. In fact, definable terms are superfluous, and only undefined terms are indispensable. (Russell [52, p. 259])

Russel's program aiming at the clarification of the basis of scientific disciplines by elimination of redundancy is driven essentially by syntactical attitude and procedures, which are natural and common in Mathematics. Lattice Analysis formalizes this syntactical reduction process precisely by pointing out the attributes (resp. objects) which are superfluous since their extension (resp. intension) can be expressed by intersection of others, so that the corresponding element is not meet-irreducible (resp. join-irreducible). See Fig. 3, where only 10 out of the 26 letters are join-irreducibles, the remaining 16 being superfluous. However, it does not say anything about the choice of the basic attributes, which at the end of the reduction process turns out to become the undefined terms (axioms). For these definitions and choices, one cannot stay in the syntaxic side:

> Qu'une science soit considérée simplement comme un système d'affirmations ou comme une totalité de certaines affirmations et de certaines activités humaines, de toutes manières l'étude du langage scientifique constitue une partie essentielle de la discussion méthodologique d'une science. (...) La sémantique du langage scientifique doit être tout simplement incluse dans la méthodologie de la science comme une de ses parties. (Tarski [33, p. 300]) (8)

One of the potential duty of Lattice Analysis and discrete methods can be of help clarifying the basis of scientific disciplines, in offering syntactical tools in order to be in a better position to evaluate and refine their semantic, and this in using the available dualities *classes/propositions*, *examples/counterexamples*, and *extension/intension*.

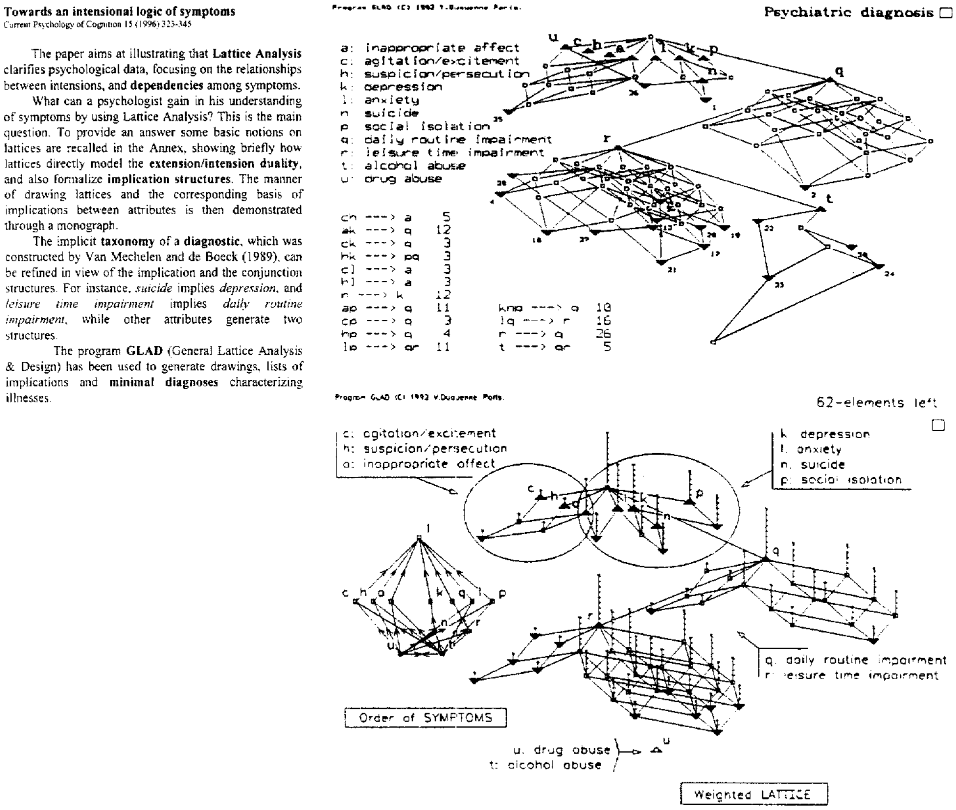
Now, due to the potentially exponential unfolding of the binary relation into the Galois lattice, it remains a major question concerning Lattice Analysis – as well as other discrete methods – which is the need of developing *approximation procedures*. Thus,

> C'est là un fait philosophique général: l'analyse ne rend jamais raison de la synthèse. (Bachelard [42, p. 101]) (9)

Hence, in this spirit, eventually as the outcome of a process consisting of several back and forth interactions between the *reality under study*, the *analysis*, and *theoretical considerations*, to be in a position to reduce the number of attributes, and thus to accept losing precision in the raw data, may appear as a fundamental prerequisite and goal of Lattice Analysis. This can be done in two ways: first, by refining the attributes' definitions, it is often possible to eliminate unnecessary precision by recombining several attributes into a single one (via Boolean expression, for example). This a priori reduction is driven by the semantic of the domain, and should be done in narrow cooperation with the user and specialist of the domain. Second, as in Fig. 6, by applying some kind of threshold for cutting the original lattice, by a more syntactical and neutral procedure. This two kinds of reductions can obviously be combined together, as for the following example, of which the remaining 11 attributes were obtained by recombination of the 23 original ones, which generated too large a lattice, quite difficult to analyse.

Fig. 7 (extracted from [66]) displays groups of patients which entered a psychiatric ward, described by 11 traits, for a small population of 30 subjects. Both drawings contain the implications $t \rightarrow r \rightarrow q$ (which express troubles in *mastering time*). The top drawing displays the Galois lattice and the basis of implications of the raw data, while the other one the *derived* lattice, *weighed* by the extensions' cardinalities (scaled in 10% of the population), which is obtained after applying a 10% *cut approximation procedure*. In this new lattice, "u" and "t" become superfluous, since they are pushed to the bottom and are no longer meet-irreducible elements. Without entering too much into the data interpretations, this approximation confirms the scaling in three levels of troubles in mastering time, and at the top, the existence of two groups of attributes that do not associate together (encircled by ellipses in the bottom drawing).

Such orders of attributes/Galois lattices will be used to reduce the complexity of the conceptual field born by all these groups, and to abstract approximate implications (dependencies, rules) between attributes. The weighed lattice can be interpreted as the union of all the *cumulated distributions* along the maximal chains between the lattice top and bottom. In order for Lattice Analysis to become a standard statistical method and not only a representation of the data, it should be grafted together with approximation procedures and more sophisticated threshold methods. Such a program intails specific difficulties: to construct a Galois lattice from a table with n attributes requires at most to scan through 2^n – eventually with drastic shortcuts; to extract from



Towards an intensional logic of symptoms
Current Psychology of Cognition 15 (1996) 323-345

The paper aims at illustrating that **Lattice Analysis** clarifies psychological data, focusing on the relationships between intensions, and **dependencies** among symptoms.

What can a psychologist gain in his understanding of symptoms by using Lattice Analysis? This is the main question. To provide an answer some basic notions on lattices are recalled in the Annex, showing briefly how lattices directly model the **extension/intension duality**, and also formalize **implication structures**. The manner of drawing lattices and the corresponding basis of implications between attributes is then demonstrated through a monograph.

The implicit **taxonomy** of a **diagnostic**, which was constructed by Van Mechelen and de Boeck (1989), can be refined in view of the implication and the conjunction structures. For instance, *suicide* implies *depression*, and *leisure time impairment* implies *daily routine impairment*, while other attributes generate two structures.

The program **GLAD** (General Lattice Analysis & Design) has been used to generate drawings, lists of implications and **minimal diagnoses** characterizing illnesses.

Fig. 7. Psychiatric diseases of a group of patients

this raw lattice a “nice” derived sub-structure requires to scan through eventually 2^{2^n}, which is at another scale.

Fig. 8 represents the description of social *practices* by *words*, concerning the treatment of poverty in New York city, at the end of past century (extracted from Mohr and Duquenne [27]). The lattice is “nearly” distributive (when removing only four elements: between *give$* and *investg*, *homeWk*), and therefore generates splitting practice/word pairs:whenever a pair *p*/*w* of *practice*/*word* is such that both are *irreducible* and that *p* is the *lower practice not below w* while *w* is the *higher word not above p* in the lattice, the pair *p*/*w* is said to be *perspective*. For instance, the pair *food*/*FALLEN* is perspective. When a lattice is distributive, there is an exact *matching* between the join- and meet-irreducible elements, each of these matching pair being perspective (and there is not any other perspectivity relationship), and each matching pair expresses a *local negation* (*p* is the *complement* of *w* in the interval that they generate) and an *exclusive union*: a block (*practices*, *words*) in the lattice is either above *p* or below *w*, but not both.

The distributive interval above *shelter*, in Fig. 8 is built on the following list of splitting pairs: *paidWk*/*NEEDY*, *investg*/*DESTITUTE*, *advise*/*INDIGENT*, *find Job*/

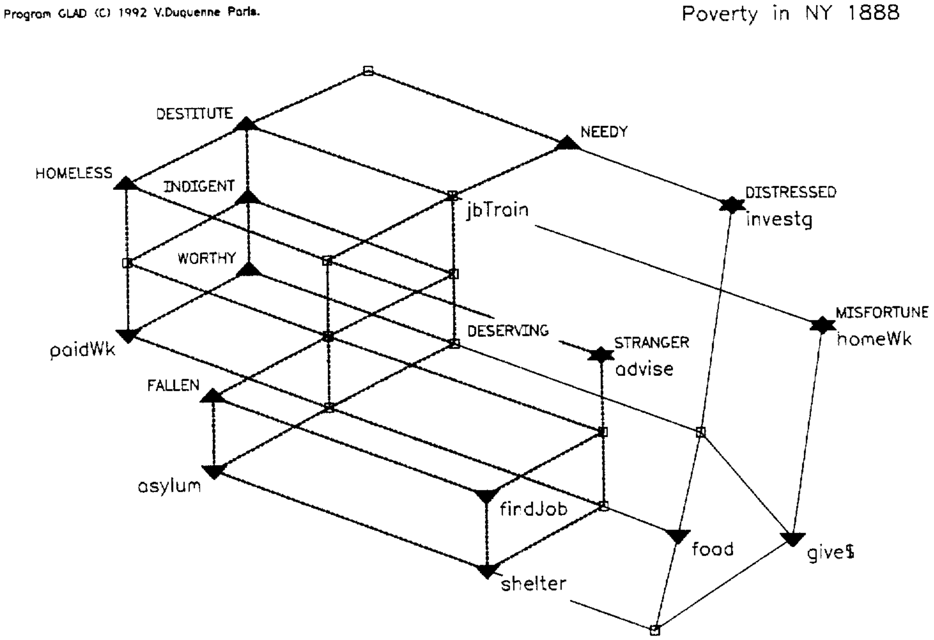


Fig. 8. The Galois lattice 'practices × words' (1888)

WORTHY, *give$/HOMELESS*, *food/FALLEN*, and finally *asylum/STRANGER*. These splits must be interpreted in connection with the order relationships between practices and words: hence, *NEEDY* applies to all the practices implied by *paidW* (*asylum*, *food*, *shelter*) but itself, and is therefore characteristic of not getting only *paidWk* but *food* or cover. When *STRANGER* applies, it cannot be to an *asylum* but a *shelter*; and when *FALLEN* applies, it is not to *food* but to *shelter*. For *HOMELESS*, give$ doesn't apply, but eventually *food*, *shelter*, *asylum*, *findJob*, or *paidWk*.

Fig. 9 illustrates another example (extracted from Schweizer [53], see also [62]). The possessions of 98 Javanese peasants have been described into 27 binary attributes as part of an anthropological research project on *Material Culture*. Lattice Analysis has been used on this set of "raw" data in order to assess the connection between *people* and *possessions*, to summarize the main co-occurrences of possessions, and to try to decipher the class structure of this village. Three local analyses have been done, focusing on the attributes defining *housing*, *livestock* and *furniture*. *Housing* alone gives limited information, *livestock* shows highly non-symmetrical associations, while the *furniture* structures the peasants along "regular" blocks that are linearly ordered along the implications: *television* → *mattress* → *bicycle*.

Most interesting in Fig. 9 is that the global analysis also shows the population as being ordered along *regular blocks* that are characterized by *housing* only: hence, the class structure is assessed by *ownership*, *access to water* (well), and the *size of the houses*, even if *livestock* and *furniture* refine this global ordered class stratification.

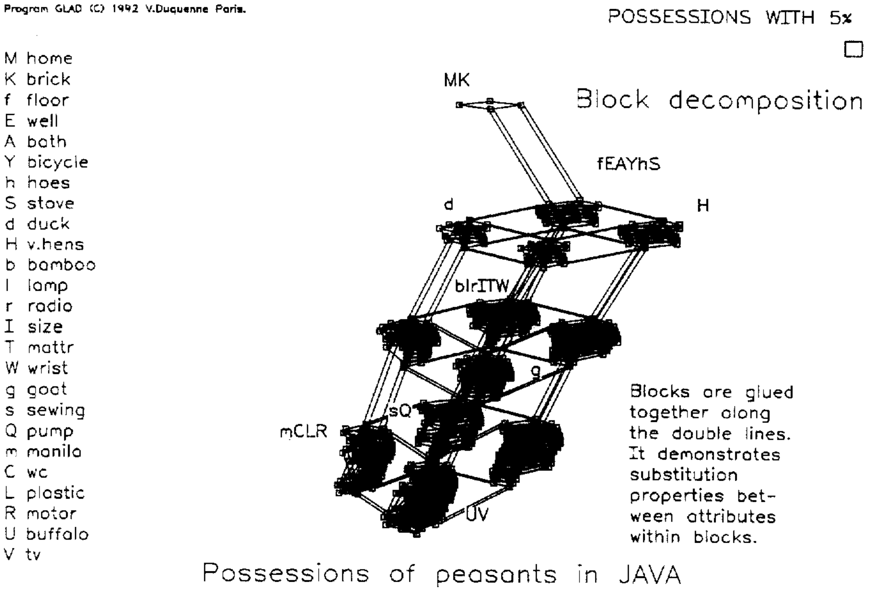


Fig. 9. Ungluing decomposition in regular blocks

These provisional conclusions should be enriched by the anthropological knowledge of the experts, taking an extensional look and coming back to the peasant descriptions.

To conclude this section, let us quote a glad user of Lattice Analysis with whom this analysis was developed, in an essay on *Discrete Structures in the Social Sciences*, which tried to evaluate precisely *what may be gained* by using it in Anthropology:

> What do we gain by applying lattice analysis to material possession data? Data on material possessions are intrinsically meaningful for studying the fundamental organizational principles and the basic scaffolding of societies. At the heart of material possessions data are logical implications (what is associated with what?) and the notion of dual ordering: what can we infer from the possessions data on the ordering of actors as well as the ordering of things? (...) This visualization of often complex data enables deeper understanding of the inherent ordering pattern and provides a spur to ethnographic interpretation. (Schweizer [53, p. 27])
>
> In a sense this discrete movement takes up the gist of older structuralisms in the social sciences, but structural analysis of today studies orderings in empirical data by using precise methods, newly implemented computer programs and thus avoids the flaws of its predecessors. (Schweizer [53, p. 4])

6. Conclusion

Since Descartes, Human beings have been thought of as a “geometrical animal”, and scientific literature has been affected by this geometrical shift. In the present case of the use of Latticial Structures in Data Analysis, notice how it accomplishes a circular return to the initial motivations of a precursor of Lattice Theory such as G. Boole:

> Au total, l'émergence d'un point de vue algébrique, objectif du mouvement de l'Ecole analytique anglaise, devait se faire contre une tradition de géométrisme, et c'est dans les termes de cette opposition également que se traduit la question notationnelle. (...) Cette affirmation de l'autonomie des méthodes purement analytiques de résolution de problèmes est le point de départ d'une véritable philosophie du symbolisme, d'une réflexion poussée sur ce que signifie raisonner dans les signes, et sur la différence entre les symboles algébriques et les figures géométriques. (Diagne [47, p. 78]) (10)

If Boole implicitly conceived his lattices to reason on formulas, using Galois lattices, for example in the Social Sciences, can now be done in order to be in a position to represent and understand the social structures. With this inversion, one joins the viewpoints of another great logician and mathematician of the past century, C.S. Peirce:

> Ce qui distingue réellement les mathématiques [selon Peirce] ce n'est pas leur objet, mais leur méthode, "qui consiste à étudier des constructions ou diagrammes". "Que telle soit bien la méthode des mathématiques, ajoute-t-il, est sans aucun doute correct; car même en algèbre le but du symbolisme est de mettre sous les yeux de l'esprit une représentation schématique des relations mises en cause dans le problème, qui pourra être étudiée comme on étudie une figure géométrique". (J.Chenu in Peirce [50, p. 66]) (11)

G. Bachelard goes further when he argues that a drawing can be more than a useful tool, it can acquire the status of a conceptual interface linking concrete/abstract levels:

> Rendre géométrique la représentation, c'est à dire dessiner les phénomènes et ordonner en série les événements décisifs d'une expérience, voilà la tâche première où s'affirme l'esprit scientifique. C'est en effet de cette manière qu'on arrive à la quantité figurée, à mi-chemin entre le concret et l'abstrait, dans une zone intermédiaire où l'esprit prétend concilier les mathématiques et l'expérience, les lois et les faits. (Bachelard [43, p. 5]) (12)

The lattices which have been generated from real observations – as in Fig. 6 and after – "put under the mind's eyes" all the relationships between the groups of patients that share the same conjunctions of attributes. Such representations should not be ultimate statements, but some kind of frames on which thinking can be fixed. By comparing sub-structures, by checking the attributes associations, the groups' cardinalities. Then, Galois lattices acquire the beginning of an existence as technical tools.

Hence, the questions which have been treated here about Latticial Structures in Data Analysis are operating in the contact area between observational and experimental disciplines on the one hand, and Mathematics on the other and in so doing there is an exchange. What are the relationships between these worlds that rarely meet? What are the needs of exchanges between them? The expressed needs of interactions?

> Contrairement aux opinions du sens commun, il est donc beaucoup plus difficile de constater des faits et de les analyser que de réfléchir ou de déduire, et c'est pourquoi les sciences expérimentales sont nées bien après les disciplines déductives, celles-ci constituant à la fois le cadre et la condition nécessaires de celles-là, mais nullement suffisants. (Piaget [51, p. 99]) (13)

> La connaissance scientifique comporte deux modes fondamentaux: l'interprétation expérimentale et la déduction algorithmique, pouvant d'ailleurs être tous deux selon les cas plus ou moins statiques ou dialectiques. En un mot, les sciences supposent des faits et des normes, et elles se chargent de découvrir ou d'élaborer les deux. (Piaget [51, p. 153]) (14)

The "historical gap" and the specific difficulties of the Natural Sciences do not imply in turn the absence of results. on the contrary. Facing the extraordinary explosion of the Natural Sciences, F. Engels already fixed as an urgent goal, last century, to order and formalize their numerous achievements. This is still a challenging program today.

> L'étude empirique de la nature a accumulé une masse si énorme de connaissances positives que la nécessité de les ordonner systématiquement et selon leur enchaînement interne dans chaque domaine de recherche séparé est devenue absolument impérieuse. (...) Mais la science de la nature, ce faisant, se transporte dans le domaine de la théorie et ici les méthodes empiriques échouent, la pensée théorique peut seule servir. (Engels [48, p. 49]) (15)

Now, if Mathematics and Logic have an autonomous development through their exchanges with the Natural Sciences, is this exchange fair and symmetric? Is there any relationship of domination or conquest, in a relationship of exteriority? Any hierarchy?

> Mais le rapport des mathématiques aux sciences de la nature n'est pas à sens unique. (...) Tout se passe comme si les mathématiques rendaient aux sciences, sous une forme élaborée, ce qu'elles ont reçu d'elles. Dans cet échange organique, a-t-on encore le droit de parler d'application ? Ne doit-on pas parler un autre langage, et dire qu'il existe entre les mathématiques et les sciences de la nature un autre rapport, un rapport de constitution - les mathématiques n'étant ni un outil, ni un instrument, ni une méthode, ni un langage au service des sciences, mais partie prenante à leur existence, à leur constitution ? (Althusser [40, p. 32]) (16)

This position could be more widespread and shared among mathematicians, experimenters and other scientists, when they are pained by their efforts to communicate.

Reaching now the conclusion of this discussion, and comforted by all these viewpoints of the literature of the Philosophy of Sciences, we are now in a better position to try making the initial thesis more precise, in claiming the following re-stucturalist thesis: *Lattices are natural and fundamental structures which are constitutive of any attempt to describe and analyse data from the Natural, Behavioral and Social Sciences, and this, either in the direction of experimental designs, or of observational dichotomic data, due to their ability to encode dualities, and to bear both thought and calculus.* Moreover, in reaction to a spreading wave of neutralism in the face of data, we also subscribe to the following complementary precision:

> Le donné est relatif à la culture, il est nécessairement impliqué dans une construction. (...) Il faut qu'un donné soit reçu. Jamais on n'arrivera à dissocier complètement l'ordre du donné et la méthode de sa description non plus qu'à les confondre l'un dans l'autre. Il y a entre ces deux termes – qui représentent pour nous l'opposition minima de l'esprit et du réel – des réactions constantes qui soulèvent des résonances réciproques. (Bachelard [42, p. 14]) (17)

Following this position, Lattice Analysis would not be established as an autonomous discipline, developing specific vocabulary and folklore, but should stay as a coordinated family of analysis practices, preserving carefully its roots in Mathematics as well as in the kernel of the disciplines with which it can be applied and further developed.

Appendix. Translations of the quotations

[*These translations are given for the convenience of the English speaking reader. They are nearly "word for word" transcriptions of the original French quotes, as neutral as is hopefully possible.*]

1. Whatever be the starting point of the scientific activity, this activity can be fully convincing only by leaving its basic domain: if it experiments, one has to reason, if it reasons, one has to experiment.

2. Following Dedekind, a great number of contemporary researchers – MM Birkhoff, Von Neumann, Glivenko, Ore and others – have built a general theory of structures (English authors call them lattices, networks) which comprise set theory, number theory, projective geometry, combinatorial topology, probability theory, mathematical logic, functional spaces, etc ... (...) The general theory of structures lays on the possibility to structure along two inverse ways a same set, and this is for us a result of philosophical importance to see this inner duality of two antisymmetric beings/things [êtres], that can be distinguished in the core of a same being, which forms the generative principle of a huge gathering of mathematical realities.

3. On the other hand, it appears that the authors could without any inconvenience have omitted the chapter on lattices, to which an important American school pays a persistent attention, despite the least interest that this theory provides in the other branches of mathematics.

4. At first sight, it seems that the purpose of propositional calculus and projective geometry are distinct, although the logical structures of these two topics have a lot of analogies. The meaning of these analogies appeared only in very recent researches within the new domain of abstract algebra.

5. The two logical interpretations. One circumstance even appears to be particularly interesting: this algebra can receive two distinct interpretations in logic, of which the parallelism is nearly perfect, whether the letters represent either concepts or propositions. With no doubt it is possible to follow Boole and Schröder in bringing back these two interpretations to only one, by considering the concepts, on the one hand, and the propositions, on the other hand, as corresponding to sets or classes: one concept determines the set of objects to which it applies (which is called its extension in logic); one proposition determines the set of cases or of moments in time where it is true (which could analogously be called its extension); then, the calculus of concepts and the calculus of propositions reduce to only one...

6. The scientific mind can be wrong in following two contradictory tendencies: attraction for singularity and attraction for the universal. At the level of conceptualisation, we will define these two tendencies as characteristic of a knowledge in comprehension and a knowledge in extension. (...) Here it would be necessary to coin a new word, between comprehension and extension, to point out this activity of the inventive scientific mind. It would be necessary that this word received a particular dynamic acceptation. Indeed, we think that the richness of scientific concept is weighed by its power of distortion [déformation]. These potentialities cannot be attached to an isolated phenomenon which would be recognized richer and richer in features [caractères], richer and richer in comprehension. This richness can neither be attached to a collection that would put together heterogeneous phenomenon, which would spread to new cases in a contingent way. The intermediary refinement will be made real [réalisé] if the refinement in extension becomes necessary, as coordinated as the refinement in comprehension. To comprise new experimental proofs, it will then be necessary to distort primitive concepts, to study their conditions of application, but essentially for incorporating these conditions of application within the very meaning of the concept.

7. In general, the discussion of simple and complex propositions throughout the Port Royal grammar and logic suggests this concept of “idea”, since propositions are described as formed by combining ideas, and complex ideas are described as based on underlying constituent propositions. In this sense, “idea” is a theoretical term of the theory of mental processes; the comprehension (i.e., intension or meaning) of an idea is the fundamental notion in semantic interpretation, and in so far as the deep structure of language is regarded as a direct reflection of a mental process, it is the fundamental notion in the analysis of thought.

8. Independent of whether a science is conceived merely as a system of statements or as a totality of certain statements and human activities, the study of scientific language constitutes an essential part of the methodological discussion of a science. (...) The semantics of scientific language should be simply included as a part in the methodology of sciences.

9. This is a general philosophical fact: analysis never renders/gives satisfaction [rend raison] for synthesis.

10. All in all, the emergency of an objective analytical viewpoint of the analytical English school would have to be done against a geometrical tradition, and this is in the terms of this opposition that the question of denotation can be translated. (...) This claim of the autonomy of purely analytical methods for solving problems is the starting point of a real philosophy of symbolism, with a thorough reflection on what reasoning on signs means, and on the distinction between algebraic symbols and geometrical figures.

11. What really distinguishes mathematics (following Peirce) is not their objects, but their methods "which consist in studying constructions and diagrams". "That this consists in the very method of mathematics – he adds – is with no doubt correct, since even in algebra the aim of symbolism is to display a schematic representation of the relationships at stake in a problem in the face of the mind's eyes, which can be studied as a geometrical figure can be".

12. To render geometric the representation, which is to say to draw the phenomenon and to order the decisive events of an experiment in series, here is the first task where the scientific mind asserts oneself. This is indeed by this way that one reaches the represented quantity [quantité figurée], in between the concrete and the abstract, in an intermediary area where the mind pretends to conciliate mathematics and the experiments, laws and facts.

13. Contrary to the commonsense opinions, it is therefore more difficult to ascertain facts and to analyze them as compared with to think or to deduct, and here is why experimental sciences were born long after deductive disciplines, the former consisting in the frame and necessary conditions of the former, but not sufficient.

14. Scientific knowledge comprises two fundamental modes: experimental interpretation and algorithmic deduction, which by the way can be more or less static or dialectic, depending on the circumstances. In one word, the sciences suppose facts and norms, and they undertake to discover and elaborate both.

15. The empirical study of nature has accumulated such an enormous mass of positive knowledge that the necessity to order them systematically and along their internal links within the scope of each separated research domain has become a pressing absolute necessity. (...) But, so doing, the science of nature moves into the domain of theory, and at this point, empirical methods fail, only theoretical thinking can be helpful.

16. But the relation between mathematics and the sciences of nature is not one way. (...) It happens to be like if mathematics gave back to sciences what they got from them, in an elaborate way. With this organic exchange, can one still speak of application? Shouldn't one speak another language, and say that another relationship exists between mathematics and the sciences of nature, a constitutive relationship, mathematics being neither a tool, nor an instrument, nor a method, nor a language at the service of the sciences, but taking part into their existence, into their constitution?

17. A datum [donné] is relative to culture, it is necessarily involved into a construction. (...) A datum must be received. One will never succeed either in completely

dissociating the order of a datum from the method for describing it, or in assimilating them. There are between these two terms -which for us represent the minimal opposition between the mind and reality- permanent reactions that rise mutual resonances.

Acknowledgement

GLAD was also granted by Maison des Sciences de l'Homme and Maison Suger, Paris. Thanks are due to the team URA 1294 for its nice welcome, to Nanou Duquenne and John Mohr for their editorial advice, and to the GLAD users for pointing out bugs and discussing the topics of Lattice Analysis. This work is the outcome of a long process, and was presented as a Thèse d'Habilitation (1996, Univ. R. Descartes, Paris), and at ORDAL96, when visiting the Computer Science Department, Ottawa University.

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