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REPORT ON THE DISSERTATION:

GÉOMÉTRIE COMBINATOIRE, THÉORIE DES NOMBRES ET GRAPHES" BY JORGE RAMÍREZ ALFONSÍN

1. Introduction

The scientific scope of this dissertation is in the field of discrete mathematics (or combinatorics), a domain with an important theoretical development in the recent decades, reinforced by its applications to computer science.

Jorge Ramírez was published more than twenty papers in leading journals of the domain, and it is the unique author of twelve of them as sole author. He has worked in problems of well-known mathematicians as Peter MacMullen and Branko Grünbaum, and he has given total or partial solutions to it.

His monograph *The Diophantine Frobenius problem* is the standard reference on the subject. In particular, he proved that this problem is $\mathcal{NP}$-hard, a result which has been conjectured for a long time.

In this review we do not attempt to give a catalogue of all the topics covered. However, we outline a general appreciation, focussed in the scientific autonomy of the candidate, impact and originality of his work.

The content of the dissertation covers some problems of the *Elementary Number Theory, Discrete and Convex Geometry* and *Graph Theory.* We will give a short and precise survey of the problems studied and the results obtained by Jorge Ramírez in the next sections.

2. Elementary Number Theory

In Chapter 1 of his dissertation Jorge Ramírez Alfonsín describes his work in elementary number theory. More precisely, the *Frobenius number* is the largest value b for which the Frobenius equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b, \tag{2.1}$$

has no solution, where the a_i are given positive integers, $b := g(a_1, \ldots, a_n)$ is an integer, and the solutions x_i are non negative integers. J.J.

Sylvester ("Question 7382." *Mathematical Questions from the Educational Times* **4**, (1884), 21) showed

$$(2.2) \qquad g(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1 = a_1a_2 - (a_1 + a_2).$$

The challenge is to find g when $n \geq 3$. Two closely related problems are:

(2.*i*) The *denumerant problem*: For an integer m, determine the number of different ways m can be written as a linear combination of the a_i with non-negative integral coefficients. This number is denoted by $d(m; a_1, \dots, a_n)$ and is called the *denumerant.*

(2.*ii*) The *gap problem* i.e., the number of positive integers which cannot be represented by the a_i. For $n = 2$ this number is $(a_1 - 1)(a_2 - 1)/2$, but for larger n the problem is open.

Jorge Ramírez interest in this subject starts with a "naive problem": *the jug wine problem.* He founds a polynomial time algorithm that solve it (see [19] in the Dissertation). He proved that the Equation (2.1) is $\mathcal{NP}$-hard, a result which has been conjectured for a long time (see [17]). This is an elegant contribution to algorithmic number theory. He proves also that in some particular cases, the denumerant $d(m; a_1, \dots, a_n)$ can be determined in polynomial time, see (see [15]). He has also proved several results in relation to the *gap problem* in numerical semigroups. Some of them use geometry by applying *Pick's theorem*, an original approach.

These works led Jorge Ramírez to write a nice and complete book on the above Frobenius problem (see [31]). The last Appendices of this book is an interesting list of open problems and conjectures.

3. Discrete and Convex Geometry

In chapter 2 Jorge Ramírez presents his work in relation with oriented matroids (see [36]). He has tackled geometric problems proposed by international well-known names, such that Peter McMullen problem. The following question of Peter MacMullen is examined by D. G. Larman (see [45]):

(P. M.) *What is the largest number $f(d)$ such that any set of $f(d)$ points in general position in $\mathbb{R}^d$ can be mapped by a permissible projective transformation onto the vertices of a convex polytope?*

Larman obtains a complete solution for $d = 2$ and 3. In these cases $f(d)$ is 5 and 7, respectively. For $d > 3$ Larman obtains $2d + 1 \leq v(d) \leq (d + 1)^2$ and conjectures that $f(d) = 2d + 1$ for all d. In (*European J. Combin.* **22** (2001), 705-708) David Forge, Michel Las Vergnas and Peter Schuchert present an example, found with the aid of a computer, which shows that $f(4) = 9$. This problem is reformulated and generalized to a problem for oriented matroids in [42]. Using the reformulating of [42], Jorge Ramírez proves that $f(d) < 2d + \lceil \frac{d+1}{2} \rceil$, the best known upper born.

The cyclic polytope $C_d(t_1, t_2, \ldots, t_n)$ of dimension d with n vertices is defined as the convex hull in Euclidean space $R^d (d \geq 2)$ of $n (n \geq d+1)$ different points of the moment curve

$$\{m(t) = (t, t^2, \ldots, t^d), t \in \mathbb{R}\}.$$

The authors focus on cyclic arrangements $A(d, n)$, the dual to cyclic polytopes. Making use a combinatorial interpretation of arrangements in terms of oriented matroids, Jorge Ramírez deduces in [7] a explicit formula for the number of cells of $A(d, n)$ having exactly p facets $(d \leq p \leq n)$. In particular is given simple proof of a theorem by Shannon (whose original proof was quite complicated) and a proof of a previously unproved "result" of B. Grünbaum.

In [13] Jorge Ramírez constructed an infinite sequence of simple arrangements of n lines in the real projective plane with $n(n-1)/3$ triangular regions (the greatest number possible). This paper answers affirmatively to an open question of B. Grünbaum, see [36].

In [14] it is proved that there are (simple) arrangements of n lines with exactly n triangles (the minimal possible), and every two triangles does not have a vertex in common. The paper [14] also answer negatively to a question of J.P. Roudneff, see [36].

Let $U_{n,r}$ be a uniform oriented matroid having as bases, $\mathcal{B}$ and circuits $\mathcal{C}$. Let $G(\mathcal{C})$ be the graph whose vertices are the elements of $\mathcal{C}$ and where two vertices are joined if they have one base in common. We say that $\mathcal{C}_1 \subseteq \mathcal{C}$ is a *covering*, of $U_{n,r}$, if for any base $B \in \mathcal{B}$ there is a circuit $C \in \mathcal{C}_1$ such that $B \subset C$. We say that $\mathcal{C}_1 \subseteq \mathcal{C}$ is a *connected covering* if $\mathcal{C}_1$ is a covering and $G(\mathcal{C}_1)$ is connected. It is easy to show that if a covering is connected then it completely determines $U_{n,r}$. In this note, we show that connectivity is not always necessary. It is proved in [3] that some disconnected coverings determine also $U_{n,r}$. Let $s(n, r)$ be the smallest number of circuits that is sufficient to determine $U_{n,r}$. Some different upper bounds for $s(n, r)$ (by using connected coverings) are presented in [12].

A spatial representation, $\alpha(G)$, of a graph G is an embedded image of G in $\mathbb{R}^3$. A set of cycles in $\alpha(G)$ can be thought of as a set of simple closed curves in $\mathbb{R}^3$, and thus they may be regarded as a link in $\mathbb{R}^3$. A recent area of research investigates the dependence (or independence) of the link types on the structure of the abstract graph G itself rather than on specific spatial representations of G. In [2], a survey of this area is presented.

Let $m = m(L)$ be the smallest positive integer such that every linear spatial representation of the complete graph with n vertices, $n \geq m$, contains cycles isotopic to link L. In [6], Jorge Ramírez shows that $m(L) > 7$, where L denotes the second simplest nontrivial link (i.e., $L = 4_1^2$. The proof uses the (combinatorial) description of the cyclic polytope in terms of oriented matroids.

Using again oriented matroids, it is proved in [11] that $m(L) = 7$, where L denotes the *trefoil* ($L = 3_1$) or its *mirror image.*

4. Graph Theory

Given an Euler tour t of an Eulerian graph G, the *spread* of the tour t, denoted by spread(t), is the the minimum distance along the tour between two repetitions of the same vertex. Let spread(G) be the maximum value of spread(t) over all Euler tours t of G. In [16] is given some bounds for spread(K_n) for certain integers n.

In [10] is investigate whether replicated paths and replicated cycles are graceful. In particular the number of different graceful labellings of the complete bipartite graph is determined.

B. Alspach [35] posed the following question:

(B.A.) *If n is odd and the integers $a_1, \cdots, a_m$ satisfy*

$$a_1 + a_2 + \cdots + a_m = \binom{n}{2}, \; 3 \leq a_i \leq n$$

[*resp. if n is even and*

$$a_1 + a_2 + \cdots + a_m = n(n-2)/2, \; 3 \leq a_i \leq n]$$

does

$$\langle C_{a_1}, \cdots, C_{a_m} \mid K_n \rangle$$

[*resp.*

$$\langle C_{a_1}, \cdots, C_{a_m} \mid K_n - F \rangle,$$

where $K_n - F$ is the complete graph from which a 1-factor has been removed] *where C_{a_i} is a cycle of length a_i?*

In [18] the answer to Alspach's question (B.A.) is proved for some families of integers a_i.

The *combinohedron*, denoted by $C(r_1, \ldots, r_m)$, is the loopless graph whose vertices are the n-tuples in which the symbol e_i appears exactly r_i times, and where an edge joins two vertices if and only if one can be transformed into the other by interchanging two adjacent entries. (The graph permutohedron is a particular case of the combinohedron).

In [4] is extended to the combinohedron some results on embeddability of the permutohedron.

5. Conclusion

In conclusion, this dissertation is an excellent contribution to the literature of mathematics. Jorge Ramírez has successfully investigated problems in different areas of combinatorics.

His book on the Frobenius diophantine problem it is the standard reference on the subject. Relating Oriented Matroid Theory and Knot Theory, he has open a new area of research, proving a mature scientific autonomy. He has established fundamental results in elementary number theory, and geometric combinatorics, and has also provided some results in Graph Theory.

J. L. Ramírez Alfonsín has become a leading specialist in the field of discrete mathematics. **I highly recommend the degree *"Habilitation à diriger des recherches"* of the University Paris VI.**

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