

PLANE TRIVALENT TREES AND THEIR PATTERNS

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ABSTRACT. Our aim is to characterize the patterns of successive distances of leaves in plane trivalent trees, and give a very short characterization of their parity pattern. Additionally, we count how many trees satisfy some sequences of patterns.

1. INTRODUCTION

This paper was motivated by Problem 11298 of American Mathematical Monthly [1], whose solution is essentially the present theorem 3, rewritten to involve the triangulations of a convex polygon instead of plane trivalent trees. Briefly, a trivalent tree whose mutual distances of leaves are all even has a number of leaves multiple of 3. We extend the result to more general patterns of distances between leaves. We also count the trees in some sequences of patterns.

2. PATTERNS

Given a plane tree, one may describe a closed walk on the tree, visiting each leaf once and each edge twice, without crossing the edges, and record the distances of successive leaves. For example the trivalent tree with 4 leaves and 2 internal nodes of Figure 1 has the pattern 2,3,2,3 (in short $[2, 3]^2$) since the walk starting at a gives successively b at distance 2 from a , c at distance 3 from b , d at distance 2 from c and at last a at distance 3 from d .

Of course, if one uses non-planted trees, drawn in a non-oriented plane, the pattern is defined only up to a rotation and a symmetry, since one may start at any leaf, and use a right-hand first or a left-hand first walk.

Different embeddings of a same tree in the plane may provide different patterns. It is the case for the trivalent tree and its two plane embedding drawn in figure 2. This proves that the pattern is associated to the plane tree, not the tree itself.

The pattern is in general not sufficient to get the structure of the tree, as shown on Figure 3, when degree 2 is allowed, or when degree 4 or more is allowed.

We will now consider only plane *trivalent* trees, that is each vertex has degree either 1 or 3.

Theorem 1. *A plane trivalent tree is described by its pattern.*

1991 *Mathematics Subject Classification.* 05C05 , 05A15.

Key words and phrases. Plane trees, counting.

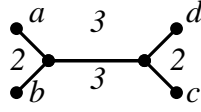


FIGURE 1. A trivalent tree and its pattern

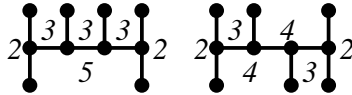


FIGURE 2. Different embeddings of a same tree

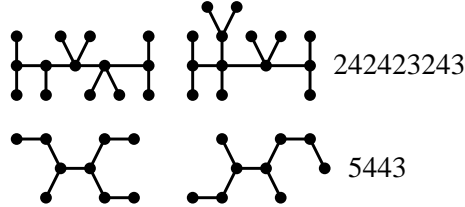


FIGURE 3. Different trees with the same pattern

Proof. We may reason by induction on the number of leaves (equivalently, the length of the pattern).

Besides the trivial case of a tree with 2 leaves (and pattern 1,1), any tree T will have a pair of leaves at maximum distance, say f_1, f_2 , and their distance d is at least 2. The neighbour g of f_1 has another neighbour at distance d from f_2 , and this neighbour must be a leaf, say f_3 . Thus the pattern of the tree T contains a 2, corresponding to the distance between consecutive leaves f_1, f_3 . Let us say it is $x2yW$. Removing f_1 and f_3 leaves a tree T' with one leaf less (since g becomes a leaf), and its pattern is $(x-1)(y-1)W$. Since the pattern of T' is sufficient to find T' , the pattern of T is sufficient to find T , indeed the difference of the patterns allows to find at which leaf of T' (up to symmetry) one has to attach two neighbours in order to recover T from T' . \square

Theorem 2. *The sum of the numbers in the pattern of a plane tree is even. For a trivalent tree having n leaves (and therefore $n - 2$ internal nodes), this sum is $4n - 6$.*

Proof. Each edge is used twice, the sum of the numbers in the pattern is just twice the number of edges. Since twice the number of edges is also the sum of the degrees, it is $n + 3(n - 2) = 4n - 6$. \square

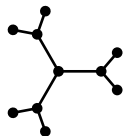


FIGURE 4. A tree with pattern $[2, 4]^3$

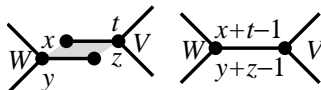


FIGURE 5. Glueing trivalent trees

3. PERIODICITY

If the pattern of a trivalent tree admits a period, say $(p)^k$ then k must divide 6. Indeed if $n = k\ell$, then the sum of the numbers in the pattern is k times the sum on a period, and also $4k\ell - 6$.

It can be 2, that means that the plane tree admits a center of symmetry, like K_2 or the graph in figure 1, and this center is the middle of an edge. It can be 3, that means that the plane tree admits a center of rotation of order 3, like $K_{1,3}$ (whose pattern is 2, 2, 2, or in short 2^3) or the graph of figure 4, whose pattern is $[2, 4]^3$, and this center is a vertex of the tree. It cannot be 6, since a plane trivalent tree cannot have a center of rotation of order 6.

4. GLUEING

Having a trivalent tree with pattern xyW and a trivalent tree with pattern ztV , one can build a trivalent tree with pattern $(x + t - 1)V(z + y - 1)W$. A picture is worth a thousand words: figure 5

5. PARITY PATTERNS

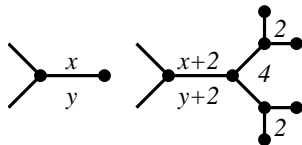
Instead of the pattern of distances, we can look at the parity pattern, that is each number in the pattern is replaced by its parity, e or o . Then it may be possible to have parity pattern with a high repetition.

For example the graph of figure 4 has a parity pattern e^6 .

The parity pattern of a trivalent plane tree cannot be any chain of e and o . First, the number of o in the parity pattern must be even. Moreover, up to the exception of K_2 (with pattern 1,1 and parity pattern oo), the pattern has to contain some 2, and therefore, the parity pattern must contain some e .

Theorem 3. e^n is a parity pattern of a trivalent plane tree if and only if 3 is a divisor of n .

Proof. If e^n is a parity pattern, then e^{n+3} is also, since from a pattern xyW , it is easy to get the pattern $(x+2), 2, 4, 2, (y+2)W$ by glueing a tree with pattern 3, 3, 2, 4, 2: see figure 6. We have seen that e^3 is the parity pattern of $K_{1,3}$. Thus e^{3m} is a parity pattern of a trivalent plane tree for any integer $m \geq 1$.

FIGURE 6. Augmenting a $[e]^n$ parity pattern

On the other hand, if we look at a trivalent plane tree with diameter $D \geq 3$, let f_1, f_2 be two leaves at distance D , then the vertex g_1 at distance 1 from f_1 has a neighbour f_3 at distance D from f_2 that is a leaf and a neighbour h at distances 2 from f_1 and $D-2$ from f_2 . Then h has a neighbour g_2 at distances 3 from f_1 and f_3 and $D-1$ from f_2 . The vertex g_2 cannot be a leaf, and its two other neighbours must be leaves, in order to satisfy the maximum distance from f_2 and the parity of distance of leaves to f_1 or f_3 . Thus the pattern is $x, 2, 4, 2, yW$, and the suppression of f_1, f_3, g_1, g_2 and the leaves adjacent to g_2 gives another trivalent tree with pattern $(x-2)(y-2)W$, and thus with parity pattern e^{n-3} . Thus, by induction, the theorem is proved

Moreover, we have seen that all trees with parity pattern e^{3n} are obtained by glueing $n-1$ times a tree with pattern $3, 3, 2, 4, 2$ to a star $K_{1,3}$. \square

Theorem 4. *A sequence of e's and o's is a parity pattern of a trivalent plane tree if and only if the 2 following conditions are satisfied*

- the number of e's is positive, except for the sequence oo .
- The value of the sequence in the group generated by e and o with relations $o^2, e^3 (eo)^2$ (isomorphic to the dihedral group on 6 elements) is the identity element.

Proof. We have already seen that oo is allowed (single edge) and eee is allowed (star with 3 rays). These are obviously the only cases with 2 or 3 leaves. They satisfy the condition. Now we will use an induction on the number of leaves.

If a sequence is allowed, we may replace a leaf v by an internal vertex adjacent to two new leaves. According to the parity of the distances of v to its neighbouring leaves, the transformation of the sequence leads to the cases $e^2 \mapsto oeo$, $eo \mapsto oe^2$, $oe \mapsto e^2o$ and $o^2 \mapsto e^3$. In each case the value of the sequence in the group is preserved. Since all trees are obtained by these transformations, the necessity is proven.

For the sufficiency, if the sequence satisfies the condition, we have to build a tree with that parity pattern.

If the sequence has only two terms, it is oo , and corresponds to K_2 ; if it has 3 terms, it is eee and corresponds to $K_{1,3}$.

If the sequence is $eeeW$ (resp. $oeoW$, resp. $oeoW$) after some rotation, we may consider that it was obtained by glueing a $K_{1,3}$ and check whether ooW (resp. $eeoW$, resp. eeW) satisfies the conditions. If yes, then we have to deal with a shorter sequence and by induction, we have won. The value of the sequence in the group is preserved, but not the presence of some e .

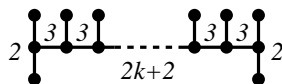


FIGURE 7. Existence of $e^3 o^{2n}$ parity patterns

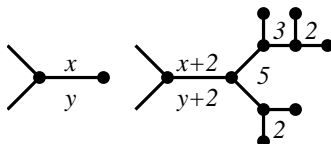


FIGURE 8. Augmenting a $[oe]^n$ parity pattern

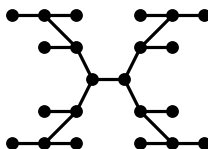


FIGURE 9. A tree with pattern $(723432)^2$

The only case where the conditions are no more satisfied is ooW , with W containing only an even number $2k$ of o 's. But $2, (2k + 2), 2, 3^{2k}$ corresponds to a tree with parity pattern $e^3 o^{2k}$, see figure 7. \square

6. EXAMPLES

Theorem 5. $[eo]^n$ is a parity pattern of a trivalent plane tree iff n is even.

Proof. This is a mere application of the previous theorem. \square

However, it can be proven in another way, that gives extra information.

If xyW is a pattern, with x even and y odd, then $(x + 2)3252(y + 2)W$ also is one. See figure 8.

Thus we have seen that any sequence $[oe]^{2n}$ is the parity pattern of a trivalent tree obtained by glueing to the tree of figure 1 a sequence of $n - 1$ copies of a tree with pattern $3, 3, 3, 2, 5, 2$. However this construction does not yield all patterns of trivalent plane trees with parity pattern $[oe]^{2n}$, since there is one with pattern $(723432)^2$ (figure 9).

7. COUNTING

It is well known that the number of plane planted trivalent trees on $n + 2$ leaves is the Catalan number $c_n = \binom{2n}{n} \frac{1}{n+1}$.

Theorem 6. *The number of planted trees with pattern e^{3n+3} is $\binom{4n+1}{n} \frac{2}{3n+2}$, the one of trees with pattern o^2e^{3n} is $\binom{4n}{n} \frac{1}{3n+1}$.*

Proof. Indeed, looking at the trees around a vertex adjacent to a leaf, we see that the generating function f of trees with parity pattern e^{3n+3} and the generating function g of trees with parity pattern $e^{3n}o^2$ satisfy $f = g^2$ and $g = 1 + xf^2$. Hence, putting $h = g - 1$, we have $x = \frac{h}{(1+h)^4}$ that allows to compute the coefficients of h and h^2 as series in x with the help of Lagrange reversion formula, and hence $g = 1 + h$ and $f = 1 + 2h + h^2$. We have $f = 1 + 2x + 9x^2 + 52x^4 + \dots$ (see sequence A069271 in [2]) and $g = 1 + x + 4x^2 + 22x^3 + \dots$ (see sequence A002293 in [2]). both with convergence radius $3^3/2^8 \simeq 0.105$. \square

Similarly the case of the parity patterns $(eo)^{2n+2}$ provides a generating function f (counting planted trees) with auxiliary functions g (counting the trees with parity patterns $o^2(eo)^{2n}$) and h (counting the trees with parity patterns $e^2(oe)^{2n}e$). They satisfy $f = gh$, $g = xfh + 1$, $h = g^2 + xf^2$. Thus $f = 1 + 4x + 33x^2 + 344x^3 + 4035x^4 + 50832x^5 + \dots$, $g = 1 + x + 7x^2 + 68x^3 + 767x^4 + 9425x^5 + \dots$ and $h = 1 + 3x + 23x^2 + 232x^3 + 2671x^4 + 33247x^5 + \dots$ and from the parametrization $f = \frac{t(t-1)^2}{(t-2)^3}$, $x = \frac{(t-2)^4}{t^2(t-1)^3}$, $g = \frac{t-1}{t-2}$, $h = \frac{t(t-1)}{(t-2)^2}$ it is easy to get the common radius of convergence $16(-5 + 26 * \sqrt{3}/9) \simeq 0.0592$ of f , g , h as series in x .

8. OTHER DEGREES

A result similar to theorem 1 holds

Theorem 7. *A tree with each internal vertex of degree k is described by its pattern.*

The proof is essentially the same: apart from the case of a single edge, each tree has some packets of $(k - 2)$ successive 2's. The pattern $x2^{(k-2)}yW$ corresponds to the tree obtained by replacing in the tree corresponding to the pattern $(x - 1)(y - 1)W$ (if it exists, of course) the leaf at distances $(x - 1)$ and $y - 1$ from its predecessor and successor by a k -star.

On the other hand, the characterisation of parity pattern (theorem 4) does not hold. The condition

- the length ℓ of the pattern satisfies $\ell \equiv 2 \pmod{(k-2)}$
- there are packets of $k - 2$ successive e 's (unless the pattern is just oo)
- the value of the sequence in the group generated by e, o with relations o^2, e^k and $(oe)^2$ (a dihedral group with $2k$ elements) is the identity element

is necessary (the proof is essentially the same), but is not sufficient. For example, the pattern e^8 does not correspond to a tree with $k = 4$.

At last, some countings have their counterparts

- the number of trees having $2 + (k - 2)n$ leaves and n internal nodes of degree k gives the generating function f with $f(x) = 1 + xf(x)^{k-1}$, and Lagrange reversion gives then easily $f(x) = \sum_0^\infty \binom{(k-1)n}{n} \frac{x^n}{(k-2)n+1}$ and more generally

$$f(x)^p = \sum_0^\infty \binom{(k-1)n+p-1}{n} \frac{px^n}{(k-2)n+p}$$

- the number of trees having all their leaves at even distances, and the degree of internal nodes k (if the distance to all leaves is even) or K (if the distance to all leaves is odd) gives a generating function f and the auxiliary function g satisfying $f(x) = g(x)^{K-1}$ and $g(x) = 1 + xf(x)^{k-1}$, hence $g(x) = \sum_{n=0}^{\infty} \binom{(k-1)(K-1)n}{n} \frac{x^n}{((K-1)(k-1)-1)n+1}$ and

$$f(x) = \sum_{n=0}^{\infty} \binom{(k-1)(K-1)n + K - 2}{n} \frac{(K-1)x^n}{((K-1)(k-1)-1)n + K - 1}$$

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- [2] N. J. A. Sloane , On-line Encyclopedia of Integer Sequences <http://www.research.att.com/~njas/sequences/index.html>

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