Basis Reduction, and the Complexity of Branch-and-Bound

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Talk outline

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- **B&B is a good algorithm after reformulation – solving almost all IPs at the root node.**

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- **B&B for Integer Programming, and why it is bad.**
- **Basis reduction.**
- **Reformulating IPs via basis reduction.**
- **B&B is a good algorithm after reformulation – solving almost all IPs at the root node.**
- **Meanings: low density knapsacks; random IPs become easier, as the coefficients become larger.**

Bounded Integer Programming (IP) Feasibility Problem

$$\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \leq \begin{pmatrix} A \\ I \end{pmatrix} x \leq \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \qquad (IP)$$

$$x \in \mathbb{Z}^n$$

Here A **is** $m \times n$**.**

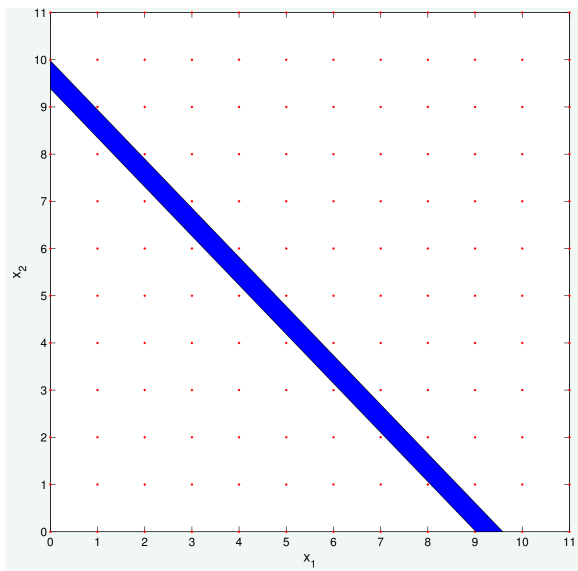
Branch-and-Bound (B&B) to Solve IP Problems

- First proposed by Land and Doig in the 60s.
- Solve the LP relaxation, and get x^*.
- If x^* has a fractional component, say x_i^*, divide the problem into subproblems by fixing x_i to its possible integer values (branching).
- Continue in this way until an integral solution is found in a subproblem, or all the subproblems are LP infeasible.
- Implemented in most software.

But: Worst case can be quite bad

In the infeasible problem below, branching either on x_1 or x_2 generates 10 nodes.

$$\begin{array}{rcl} 460 & \leq & 51x_1 + 49x_2 & \leq & 489 \\ 0 \leq & & x_1, x_2 & & \leq 10 \\ & & x_1, x_2 \in \mathbb{Z} & & \end{array}$$



Theoretically efficient methods for IP

In a sense, the best we can ask for is: polynomial running time, when n is fixed. This is achieved by:

- Lenstra's algorithm (1983);
- Kannan's algorithm (1987);
- Generalized basis reduction of Lovász and Scarf (1992).

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First two need to

- round the polyhedron (apply a transformation to make it look spherical);
- then create a small number of subproblems using basis reduction;
- do this recursively on the lower dimensional subproblems.

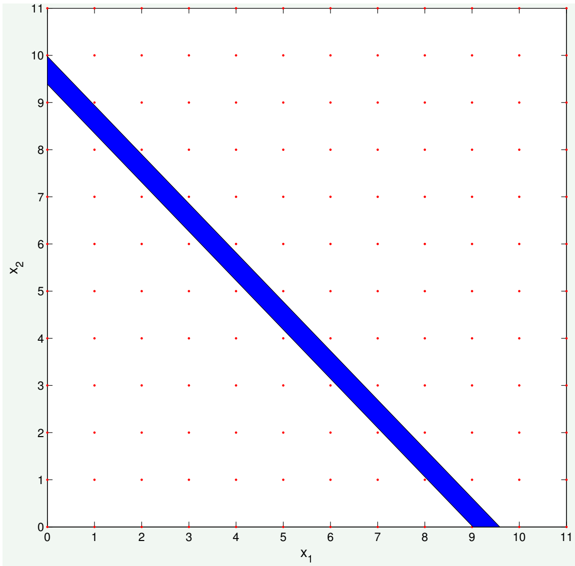
Theoretically efficient methods for IP

Generalized basis reduction: no rounding, but needs to solve linear programs to find a good branching direction.

Integer optimization w.o. binary search: Eisenbrand 2003; Eisenbrand, Laue, 2005.

We can even count the number of solutions in polynomial time, when n is fixed: Barvinok (1994); Dyer-Kannan (1997); De Loera et al (2005); Koeppe (2006).

On previous example



Lenstra's and Kannan's algorithms would round; GBR would find (by solving LPs) the branching direction $x_1 + x_2$.

What is Basis Reduction?

For rational matrix B, basis reduction (BR) finds unimodular U ($\Leftrightarrow$ U integral & $\det U = \pm 1$) such that the columns of BU are short and nearly orthogonal.

Example $$B = \begin{pmatrix} 289 & 18 \\ 466 & 29 \\ 273 & 17 \end{pmatrix}, \; U = \begin{pmatrix} 1 & -15 \\ -16 & 241 \end{pmatrix}, \; BU = \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 2 \end{pmatrix}.$$

Variants: LLL (Lenstra, Lenstra, Lovász) Reduction, KZ (Korkin-Zolotarev) Reduction, Reciprocal KZ (RKZ) Reduction.

A simpler way to use basis reduction to solve (IP)

Preprocess the problem once to make the columns of the constraint matrix reduced;

Just feed the preprocessed problem to a regular IP solver (which will use branch-and-bound).

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Nullspace reformulation: Aardal-Hurkens-Lenstra '98; Aardal-Bixby-Hurkens-Lenstra-Smeltink '00; Louveaux-Wolsey '02; For equality constrained problems, i.e. when $\ell_1 = w_1$.

Rangespace reformulation: Krishnamoorthy-P. 2005. For general IPs.

Rangespace reformulation of (IP)

$$\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \leq \begin{pmatrix} A \\ I \end{pmatrix} x \leq \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \qquad \text{(IP)} \longrightarrow$$

$$x \in \mathbb{Z}^n$$

$$\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \leq \begin{pmatrix} A \\ I \end{pmatrix} Uy \leq \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \qquad \text{(IP-R)}$$

$$y \in \mathbb{Z}^n$$

where U makes the columns of the constraint matrix reduced.

We use either RKZ- or LLL-reduction.

Nullspace reformulation of (IP)

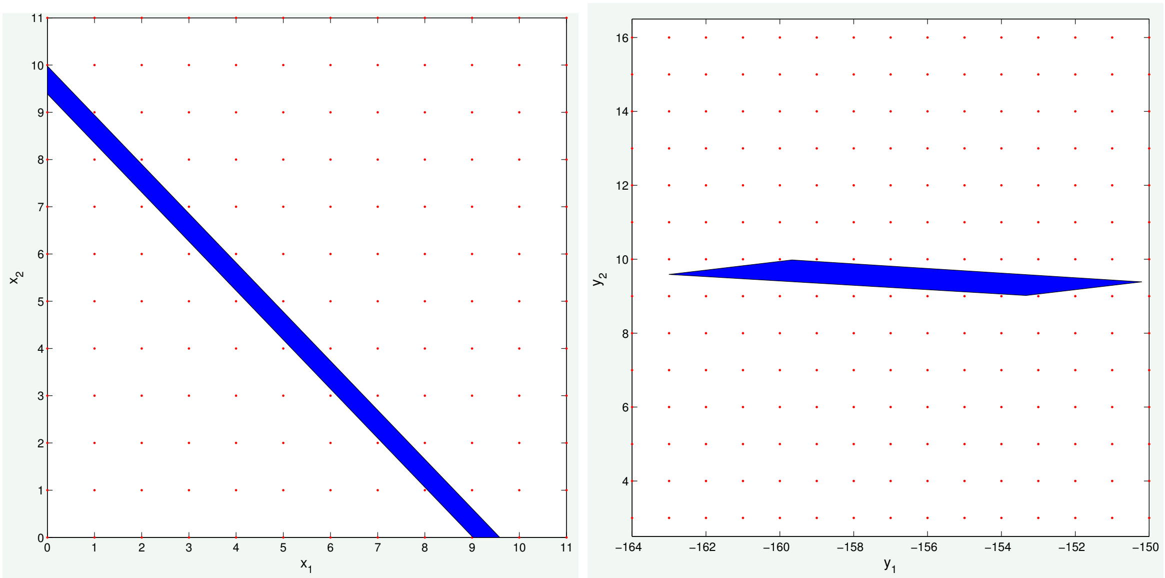
$$\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \leq \begin{pmatrix} A \\ I \end{pmatrix} x \leq \begin{pmatrix} \ell_1 \\ w_2 \end{pmatrix} \qquad \text{(IP)} \longrightarrow$$
$$x \in \mathbb{Z}^n$$

$$\ell_2 \leq B\lambda + x_0 \leq w_2 \qquad \text{(IP-N)}$$
$$\lambda \in \mathbb{Z}^{n-m},$$

where B **is a reduced basis of** $\{x \in \mathbb{Z}^n | Ax = 0\}$ **and** $x_0 \in \mathbb{Z}^n$ **satisfies** $Ax_0 = \ell_1$.

Again, we use either RKZ- or LLL-reduction.

Something nice happens when we do this (on previous example)



Analysis, for knapsack problems, assuming $a = \lambda p + r$, with λ large: Krishnamoorthy-P, 2005 (paper: Discrete Optimization, 2009).

Main point:

$$\text{width(last variable, reformulation)} = \text{width}(p, \text{original})$$

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Goal of the analysis

Given $0 < \epsilon < 1$.

If $M >$ function of $\epsilon, n, m, (\ell_1, \ell_2), (w_1, w_2)$,

then for all but an ϵ fraction of A matrices with coefficients from $\{1, \ldots, M\}$

(IP-R) solves with at most one node.

Terminology

Def: **reverse B&B** $\Leftrightarrow$ **B&B branching on** $y_n, y_{n-1}, \ldots$.

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Def: B&B solves an IP at the rootnode $\Leftrightarrow$
at every level of the tree there is at most one node.

1st ingredient

Consider

$$\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \leq \begin{pmatrix} A \\ I \end{pmatrix} U y \leq \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \qquad \text{(IP-R)}$$

$$y \in \mathbb{Z}^n$$

Lemma 1: **When we branch on $y_n, y_{n-1}, \ldots, y_1$ in this order, the total number of B&B nodes on the level of y_i is at most**

$$\left(\left\lfloor \frac{\|(w_1, w_2) - (\ell_1, \ell_2)\|}{\|b_i^*\|} \right\rfloor + 1 \right) \ldots \left(\left\lfloor \frac{(w_1, w_2) - (\ell_1, \ell_2)\|}{\|b_n^*\|} \right\rfloor + 1 \right),$$

where the b_i^* are the Gram-Schmidt orthogonalization of the constraint matrix.

2nd ingredient: when the $\|b_i^*\|$ are large

Lemma 2 $L :=$ lattice generated by columns of $\begin{pmatrix} A \\ I \end{pmatrix}$.

Lagarias, Lenstra, Schnorr, 1990: If $\begin{pmatrix} A \\ I \end{pmatrix} U$ is RKZ reduced, with Gram-Schmidt vectors $b_1^*, \ldots, b_n^*$, then

$$\|b_i^*\| \geq \frac{1}{\gamma_i} (\text{length of shortest vector in } L),$$

where $\gamma_i \leq 0.2i$ for $i \geq 10$.

3rd ingredient: when the shortest vector in L is long

Lemma 3 $L :=$ lattice generated by columns of $\begin{pmatrix} A \\ I \end{pmatrix}$, $k \in \mathbb{Z}$.

For all, but at most a fraction of

$$\frac{(2k+1)^{n+m}}{M^m}$$

of $A \in \{1, \ldots, M\}^{m \times n}$ the length of the shortest vector in L is at least k.

Main Theorem

Let $0 < \epsilon < 1$. Assume that RKZ reduction is used, and

$$M > \frac{(2\gamma_n \parallel (w_1, w_2) - (\ell_1, \ell_2) \parallel + 1)^{1+n/m}}{\epsilon^{1/m}}.$$

Then for all, but at most a fraction of ϵ of $A \in \{1, \ldots, M\}^{m \times n}$ reverse B&B solves (IP-R) at the rootnode.

Variants

- **Analysis for nullspace reformulation. It only applies when $\ell_1 = w_1$, but better bound on M.**

- **Analysis with LLL reduction. Weaker bound on M, but the reformulations are polynomial time computable.**

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Subset sum problem:

$$ax = \beta, \quad x \in \{0, 1\}^n. \quad (*)$$

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Corollary of this work: For almost all a (*) can be solved in polynomial time using branch-and-bound on the LLL-reformulations.

Meaning of Main Theorem 1

Similar result, generalizing Furst, Kannan (1987) using the RKZ reformulations,

when $M \geq 2^{cn \log n}$ (density $\leq 1/(\log n)$).

Meaning of Main Theorem 2

Practical view: an RKZ basis is easy to compute in practice if $n \leq 100$.

The number $(2k+1)^{\ell}$ is an upper bound on the number of vectors in $\mathbb{Z}^{\ell}$ with norm $\leq k$.

Using an exact count, for equality constrained, binary problems, with the RKZ-nullspace reformulation we get:

n	m	bad fraction	M
30	20	10%	31
50	20	10%	1846
50	30	10%	93

In other words

Consider the family of IPs

$$\boldsymbol{Ax = b}$$
$$\boldsymbol{x \in \{0,1\}^n}$$

with $\mathbf{50}$ variables, $\mathbf{20}$ constraints, coefficients of $\boldsymbol{A}$ from $\{\mathbf{1,\dots,1846}\}$.

The RKZ-nullspace-reformulation solves at the rootnode for $\mathbf{90\%}$ of the instances.

If we have $\mathbf{30}$ constraints, then same result holds, if coefficients are from $\{\mathbf{1,\dots,93}\}$.

Meaning of Main Theorem 3

Consider marketshare problems (Cornuéjols and Dawande)

$$\begin{aligned} Ax &= b \\ x &\in \{0,1\}^n \end{aligned}$$

and the relaxed versions

$$\begin{aligned} b - e \leq Ax &\leq b \\ x &\in \{0,1\}^n \end{aligned}$$

where $A \in \{1, \ldots, M\}^{5 \times 40}$, $b = \lfloor Ae/2 \rfloor$.

Aardal, Bixby, Hurkens, Lenstra, Smeltink (1998): the nullspace reformulations are much easier to solve by commercial solvers, than the original ones.

In the above references, $M = 100$ is used.

Meaning of Main Theorem 3

According to the theory, the reformulations should get easier, as M grows.

Results of a computational experiment: avg number (for 12 instances) of nodes to solve rangespace reformulation of inequality, and nullspace reformulation of equality constrained problems. MIP solver: CPLEX 9.1.

M	EQUALITY	INEQUALITY
100	17531.92	38884.92
1000	1254.42	22899.67
10000	200.83	1975.67

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- **B&B is efficient from a computational complexity viewpoint, after (IP) has been reformulated:**

- **For most of the instances, if coefficients are drawn from $\{1, \ldots, M\}$ for a large enough M, then reformulated problem solves at the root.**

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- **for small n and m even small values for M suffice;**
- **reformulations of random integer programs become easier as the coefficients grow.**

Papers

- Krishnammorthy, P: Column Basis Reduction and Decomposable Knapsack Problems, Discrete Optimization 2009
- P, Tural, Wong: Basis Reduction and the Complexity of Branch-and-Bound, SODA 2010
- P, Tural: Basis Reduction Methods, Encyclopaedia of Operations Research and Management Science
  (Survey on Lenstra's algorithm, Kannan's algorithm, and reformulation methods, with exercises)

Thank you!