On the Existence of Short Admissible Pivot Sequences for Feasibility Problems and Linear Optimization (LO)

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Consider
The Primal Feasibility Problem:

$$Ax = b, \quad x \geq 0$$

$A : m \times n$ matrix, $\text{rank}(A) = m$, $b \in \mathbb{R}^n$, $x \in \mathbb{R}^n$.

Problem: An arbitrary basis B and a feasible solution x^* is given.
Find a short admissible pivot sequence from B to a feasible basis.

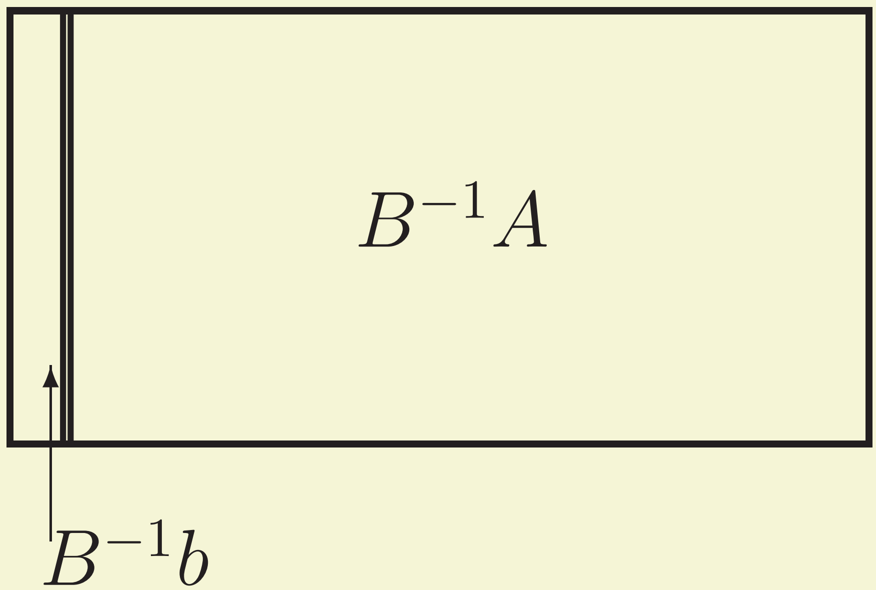
- **History:**
  - Hirsch conjecture
  - Admissible pivot methods,
  - Criss-cross methods,
  - Lüthi, Fukuda, Namiki existence under nondegeneracy
- **Admissible pivots**
- **Short admissible pivot sequence exists!**
- **Complexity: at most $n - m$ pivots**

Dual Feasibility Problem: at most m pivots
Extension to LO: at most n pivots

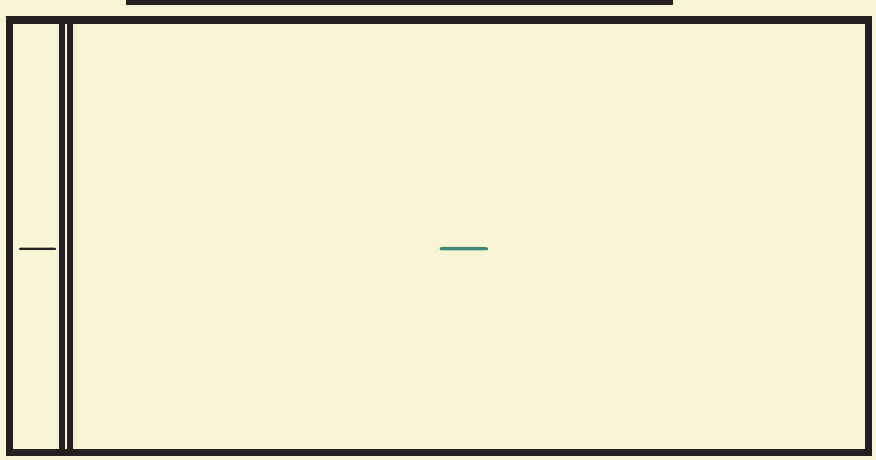
Basis B is given: $B^{-1}b = B^{-1}Ax$.

Basis solution: $x_B = B^{-1}b$, $x_N = 0$.

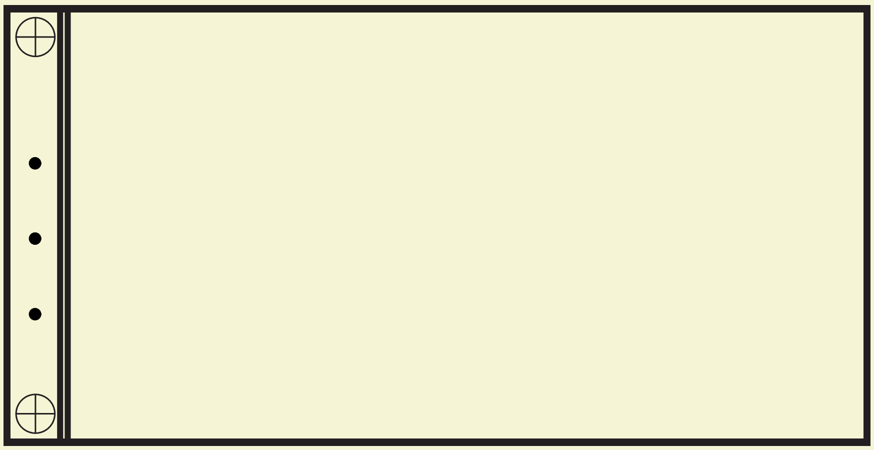
Pivot tableau



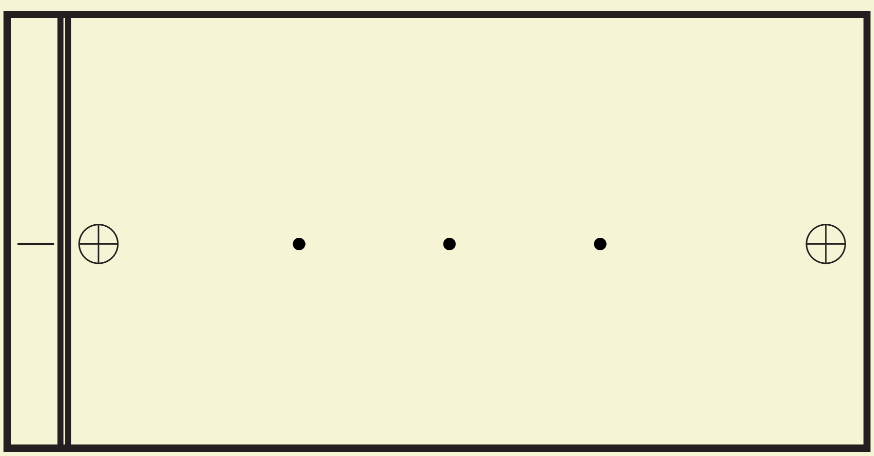
Admissible pivot



Feasible Tableau



Infeasible Tableau

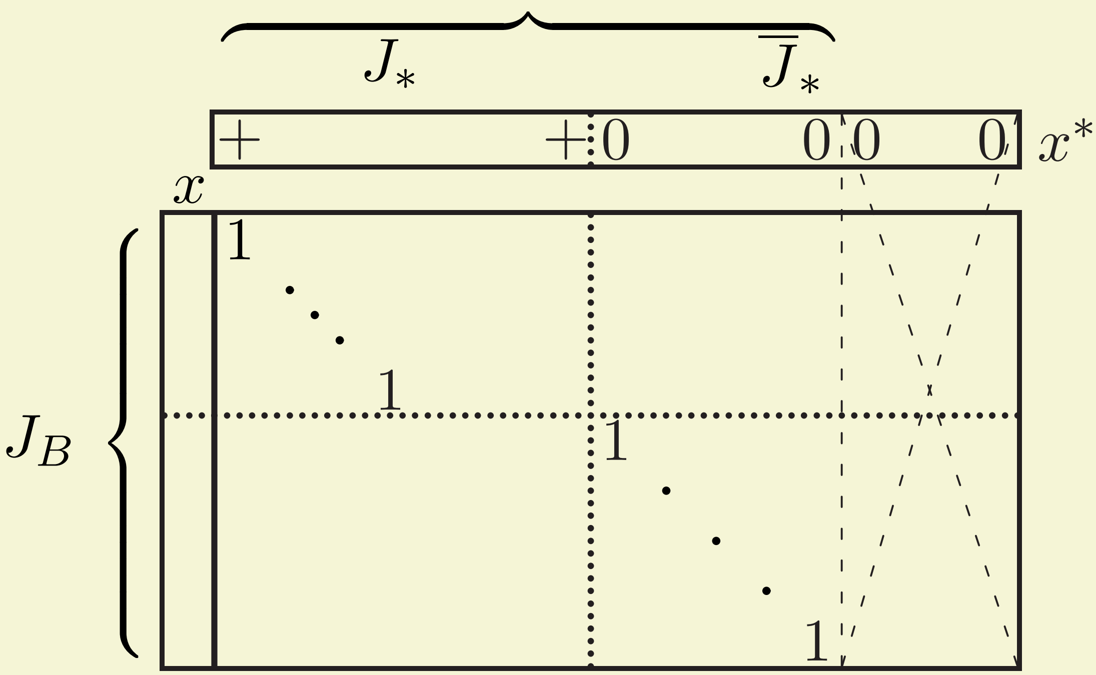


If we assume that a feasible solution x^* is given, then the latter tableau cannot occur.

Let $J = \{1, ..., n\}$, $J_* = \{i \,|\, x_i^* > 0\}$,
$J_B = \{i \,|\, i \in \text{ basis }\}$, and $I = J_* \cup J_B$.
Remove the variables not in I. Then

$$A_I x_I = b, \qquad x_I \geq 0 \qquad \text{is feasible.}$$

$$I := J_* \cup J_B$$



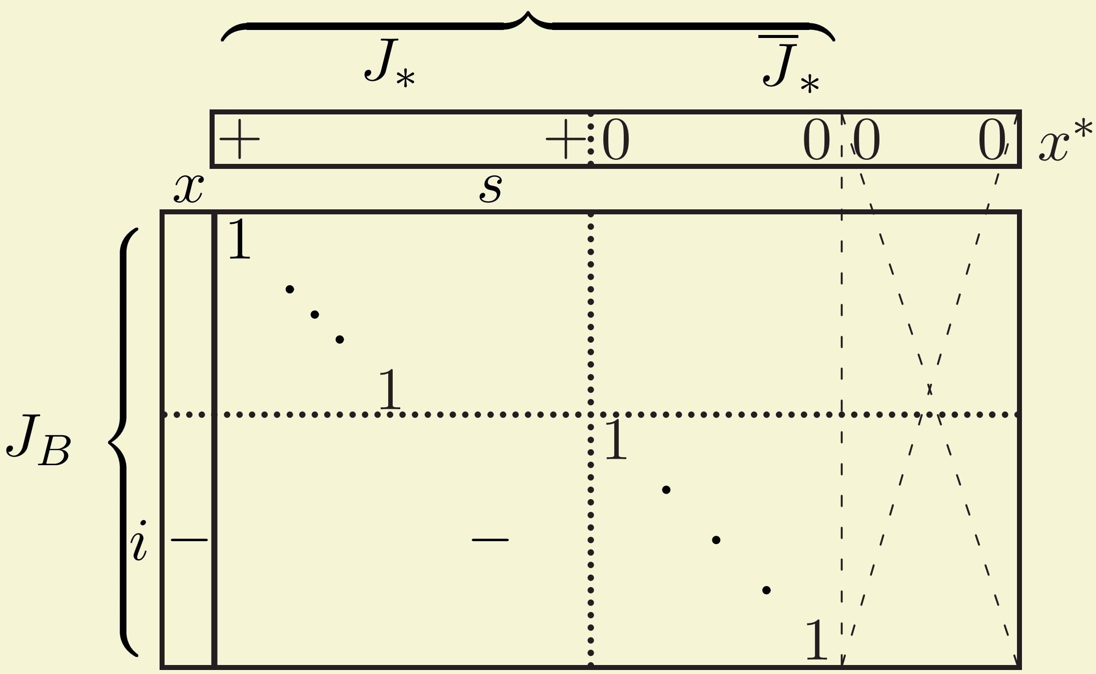
We have the following cases:

If $x_B \geq 0$, then a
feasible basis solution is found.

If $x_i \geq 0$, for all $i \in J_B \setminus J_*$,
then reduce J_*;
else reduce I.

There is an $i \in J_B \setminus J_*$ such that $x_i < 0$.
Then there is an $s \in J_* \setminus J_B$ such that
(i, s) is admissible.
(True, because $A_I x_I = b, \;\; x_I \geq 0$ is feasible.)

$$I := J_* \cup J_B$$



Do pivot at (i, s);
let $J_B := J_B \cup \{s\} \setminus \{i\}$; $I := I \setminus \{i\}$.

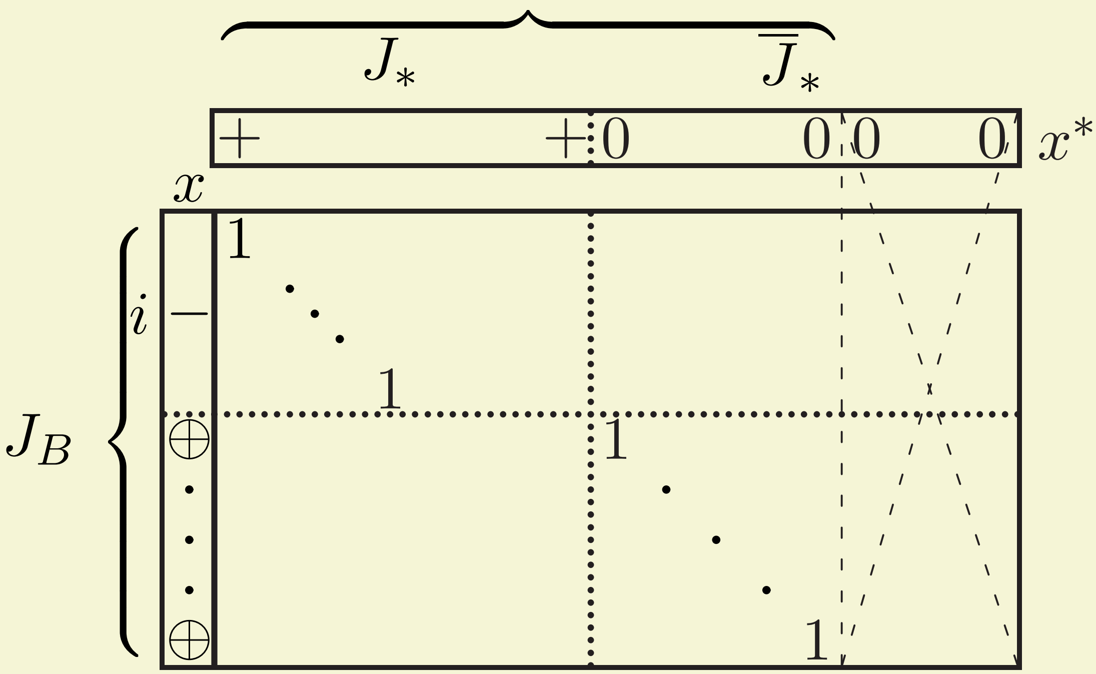
Restart the algorithm.

Note: By one pivot $|I|$ decreases by one.

Here x^* does not change, only the basis.

Here $x_i \geq 0$ for all $i \in J_B \setminus J_*$,
but there is an $i \in J_B \setminus J_*$ such that $x_i < 0$.
We eliminate a coordinate of x^*.

$$I := J_* \cup J_B$$



Let $\lambda := \min\left\{ \frac{x_i^*}{x_i^* - x_i} \mid x_i < 0 \right\} = \frac{x_r^*}{x_r^* - x_r}$.
Then we have
$x^* := \lambda x + (1 - \lambda) x^* \geq 0$ and $A_I x_I^* = b$.
Restart the algorithm
with the new x^*, $I \subseteq I \setminus \{r\}$.
Note: This way $|I|$ decreases by one.
We can always reduce I by using $x_i < 0$.

Theorem 1 *Let us consider the problem*

$$Ax = b, \qquad x \geq 0.$$

Let a feasible solution x^* and an arbitrary basis B be given. Then there is

- *a path of basis solutions initiated at B,*
- *leading to a feasible basis and*
- *connected by admissible pivots.*

The length of path is at most $n - m$.

Proof: The above presented algorithm produces such an admissible pivot path.

At the step Reduce J_* no pivots, just elimination is made.

At the step Reduce I the cardinality of I is reduced by one.

At initialization the cardinality of I is at most n, at termination I contains at least m elements, thus we need at most $n - m$ pivots to find a feasible basis. •

Note: The information that a feasible solution is known was heavily used. Basis identification in IPMs.

The algorithm does not give a strongly polynomial algorithm to solve the feasibility problem.

$$A^T y + s = c, \qquad s \geq 0$$

Basis (B) solution: $y = B^{-T} c_B, \quad s = c - A^T y.$

Pivot tableau

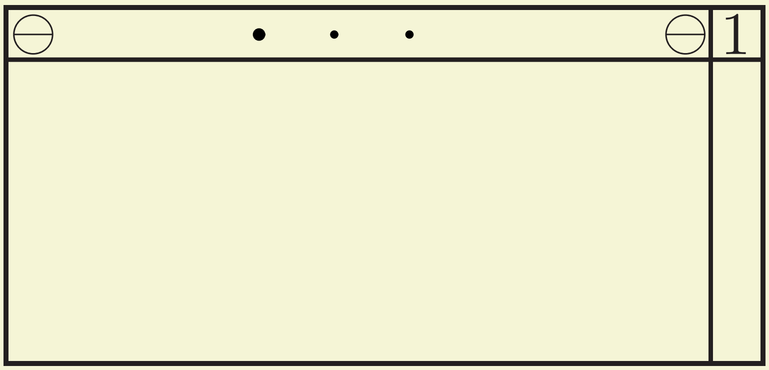
$-s = c_B^T B_{-1} A - c$	1
$B^{-1} A$	

Dual admissible pivot

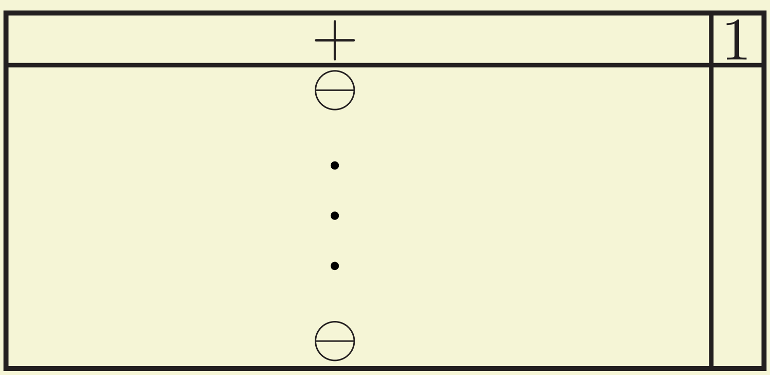
+	1
+	

Terminal Tableaus

Dual feasible tableau



Dual infeasible tableau

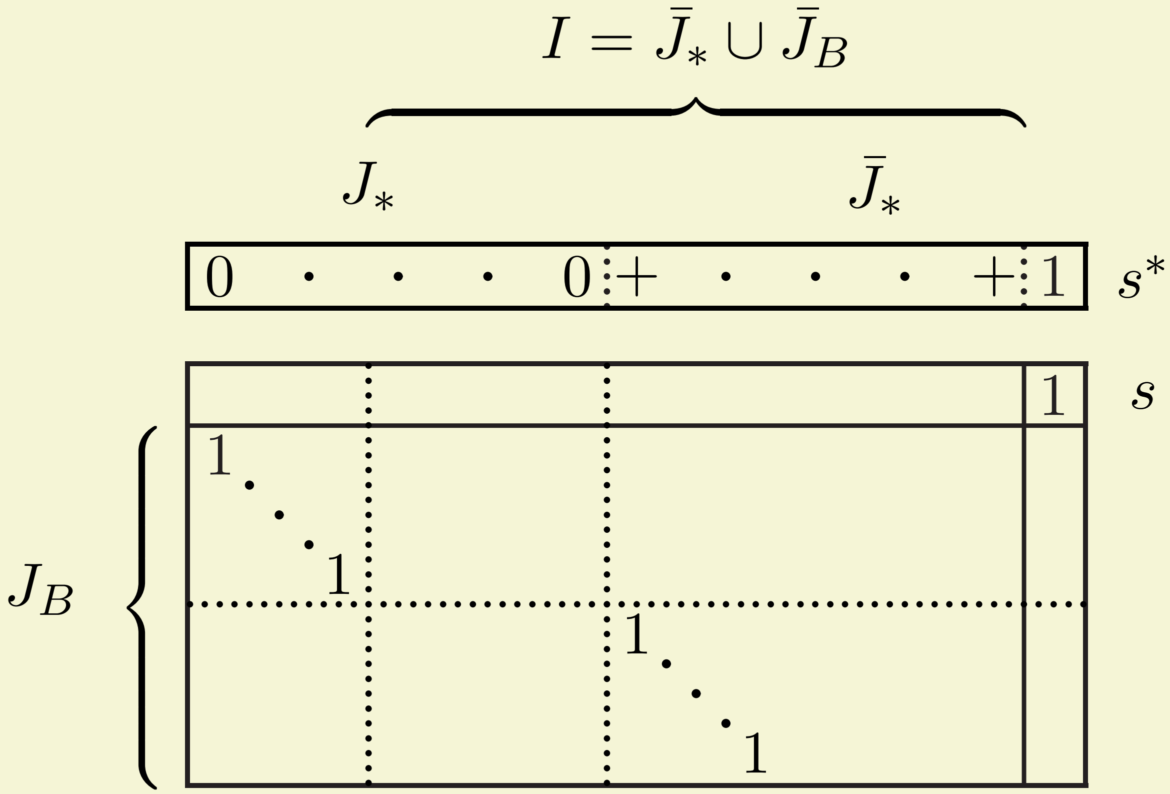


If we assume that a feasible solution (y^*, s^*) is given, then the latter tableau cannot occur.

Let (y^*, s^*) feasible be given.

Let $J = \{1, ..., n\}$, $J_* = \{i \,|\, s^*_i = 0\}$,

$J_B = \{i \,|\, i \in \text{ basis }\}$, and $I = \bar{J}_* \cup \bar{J}_B$.



We have the following cases:

If $s \geq 0$, then a

dual feasible basis solution is found.

If $s_i \geq 0$, for all $i \in \bar{J}_B \setminus \bar{J}_*$,

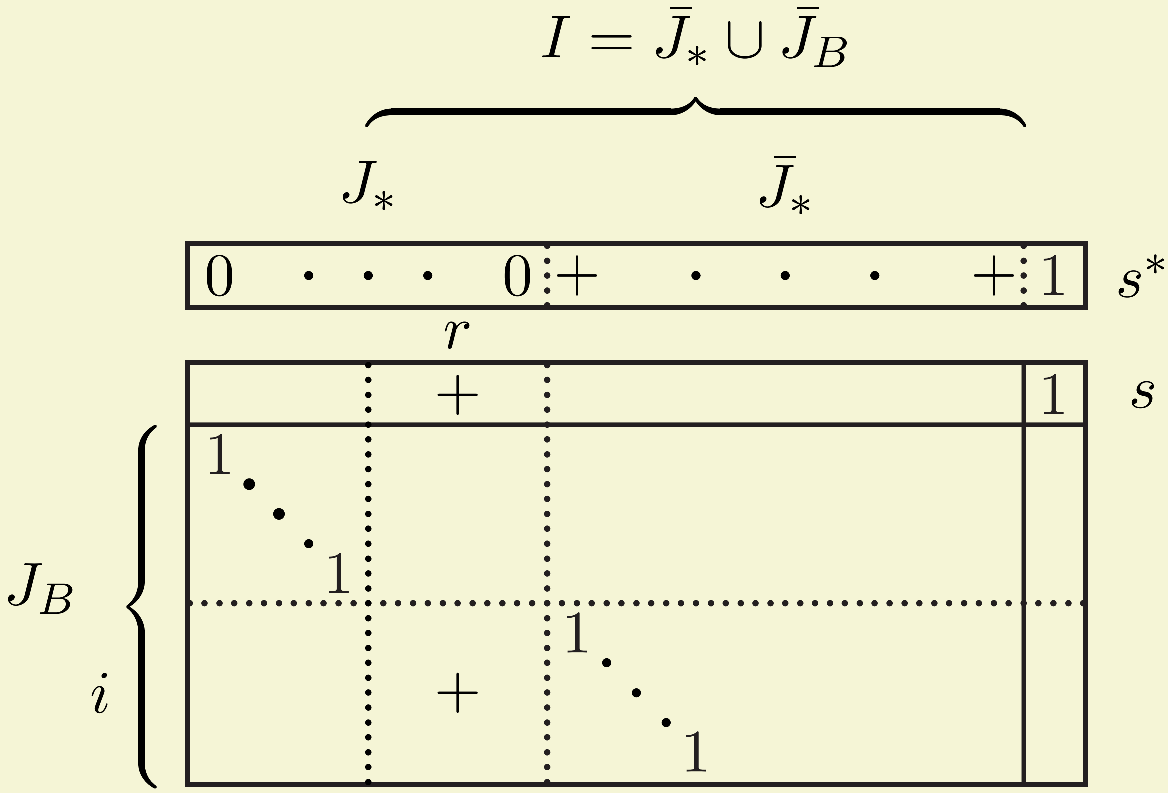
then reduce $\bar{J}_*$;

else reduce I.

There is an $r \in \bar{J}_B \cap J_*$ such that $s_r < 0$.
Then there is an $i \in J_B \setminus J_*$ such that

(i, r) is admissible.

(True because $A^T y + s = c, \quad s \geq 0$ is feasible.)



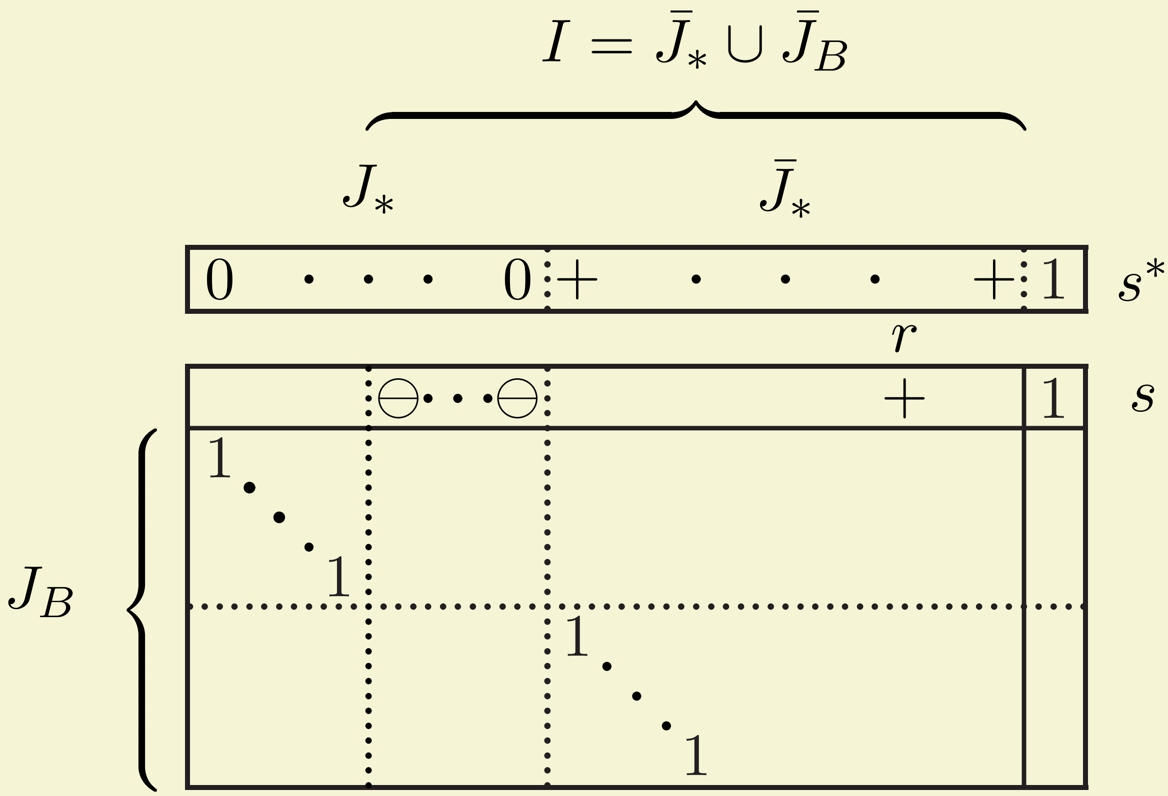
Do pivot at (i, r);

let $J_B := J_B \cup \{r\} \setminus \{i\}$; $I := I \setminus \{r\}$.

Restart the algorithm.

Note: By one pivot $|I|$ decreases, $|J_B \cap J_*|$ increases by one. Here s^* does not change.

Here $s_i \geq 0$ for all $i \in \bar{J}_B \setminus \bar{J}_*$,
but there is an $i \in \bar{J}_B \setminus J_*$ such that $s_i < 0$.
We eliminate a coordinate of s^*.



Let $\lambda := \min\left\{ \frac{s_i^*}{s_i^* - s_i} \,\middle|\, s_i < 0 \right\} = \frac{s_r^*}{s_r^* - s_r}$.
Then we have $A^T y^* + s^* = c$ and $s^* \geq 0$ with
$(y^*, s^*) := \lambda(y, s) + (1 - \lambda)(y^*, s^*)$.
Restart with the new s^*, $I \subseteq I \setminus \{r\}$.

Note: This way $|I|$ decreases by one.
We can always reduce I by using $s_r < 0$.

Theorem 2 *Let us consider the problem*

$$A^T y + s = c, \qquad s \geq 0.$$

Let a feasible solution (y^*, s^*) *and an arbitrary basis* B *be given. Then there is*

- *a path of basis solutions initiated at* B,
- *leading to a (dual) feasible basis and*
- *connected by admissible pivots.*

The length of path is at most m.

Proof: The algorithm peresented above produces such an admissible pivot path.

At the step Reduce $\bar{J}_*$ no pivots, just elimination is made.

At the step Reduce I the $|J_B \cap J_*|$ is increased by one.

At initialization $|J_B \cap J_*|$ is at least 0,

at termination $|J_B \cap J_*|$ is at most m, thus

we need at most m pivots to find a feasible basis. •

Note: The information that a feasible solution is known was heavily used. Basis identification in IPMs.

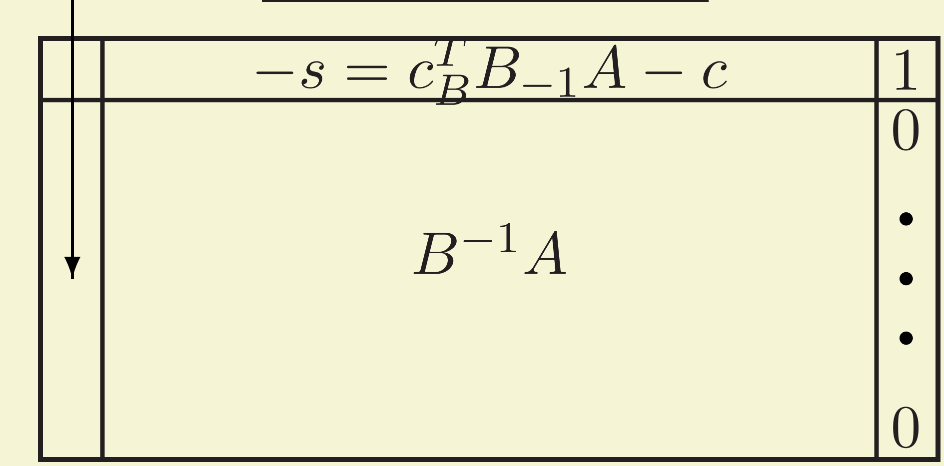
The algorithm does not give a strongly polynomial algorithm to solve the dual feasibility problem.

$$\min\{\, c^T x \mid Ax = b, \qquad x \geq 0$$
$$\max\{\, b^T y \mid A^T y + s = c, \qquad s \geq 0$$

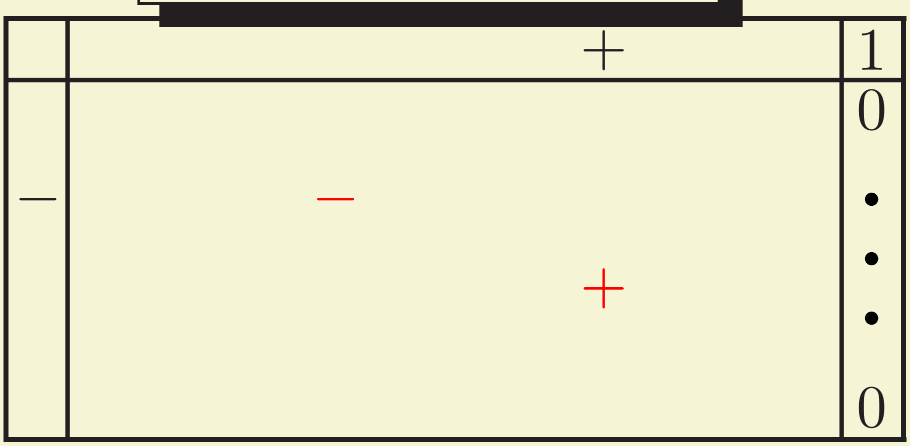
Basis (B) solution:

$x = B^{-1}b,\ Ax_B = b; \qquad y = B^{-T}c_B,\ s = c - A^T y.$

Pivot tableau

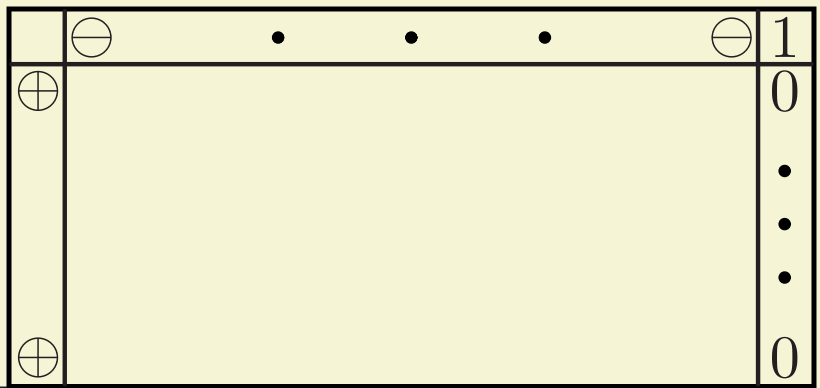


Admissible Pivots

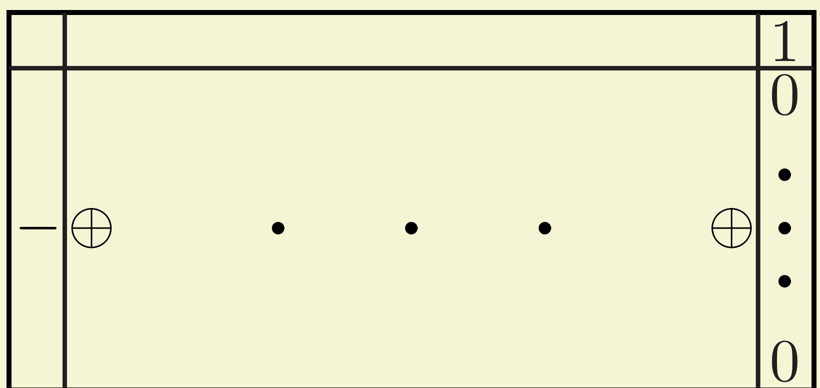


Terminal Tableaus

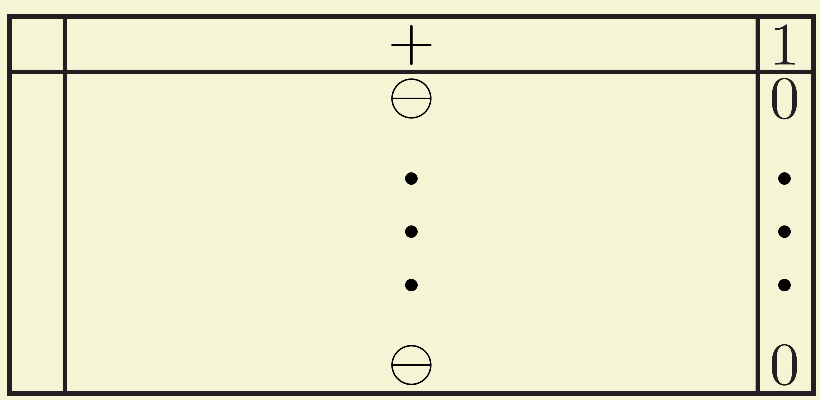
Optimal tableau



Primal infeasible tableau



Dual infeasible tableau

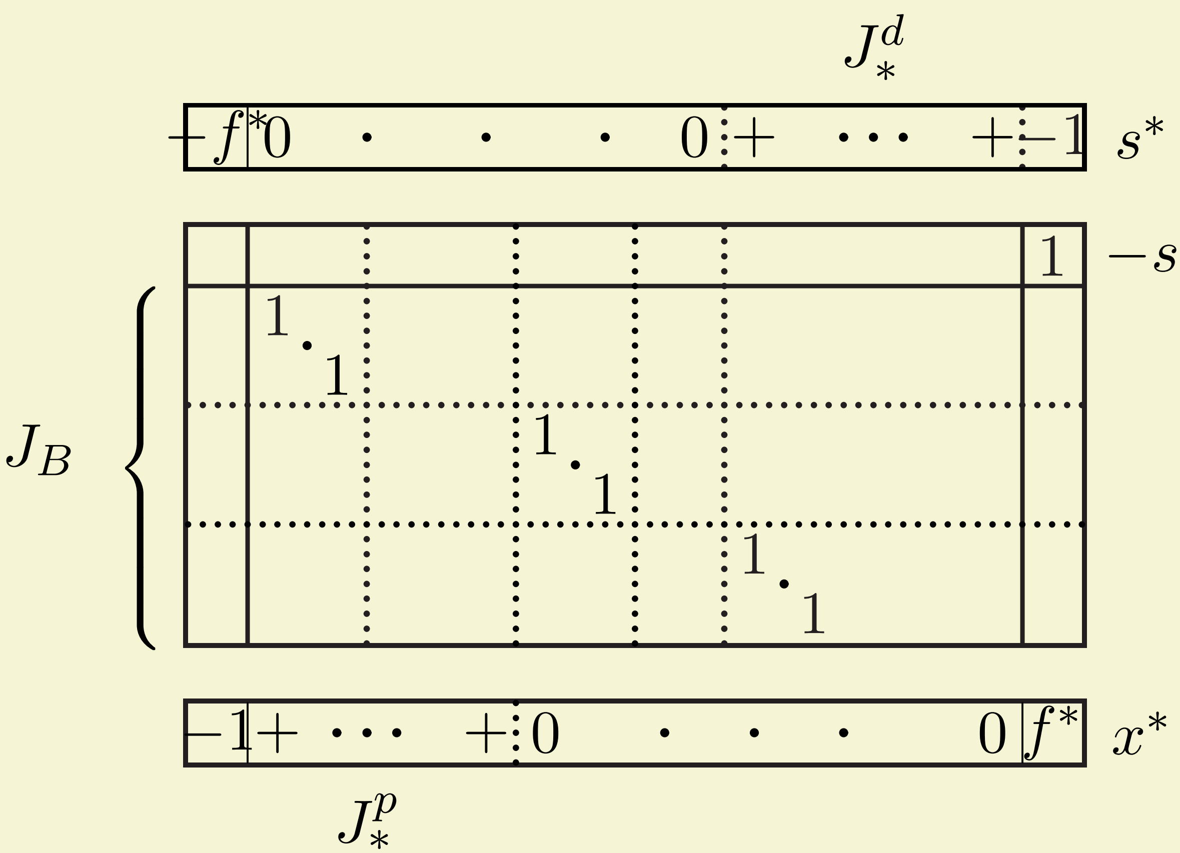


Let x^* and (y^*, s^*) optimal solutions be given.

Let

$J = \{1, ..., n\}$, $J_*^p = \{i \,|\, x_i^* > 0\}$, $J_*^d = \{i \,|\, s_i^* > 0\}$,

$J_B = \{i \,|\, i \in \text{ basis } \}$.



We have the following cases:

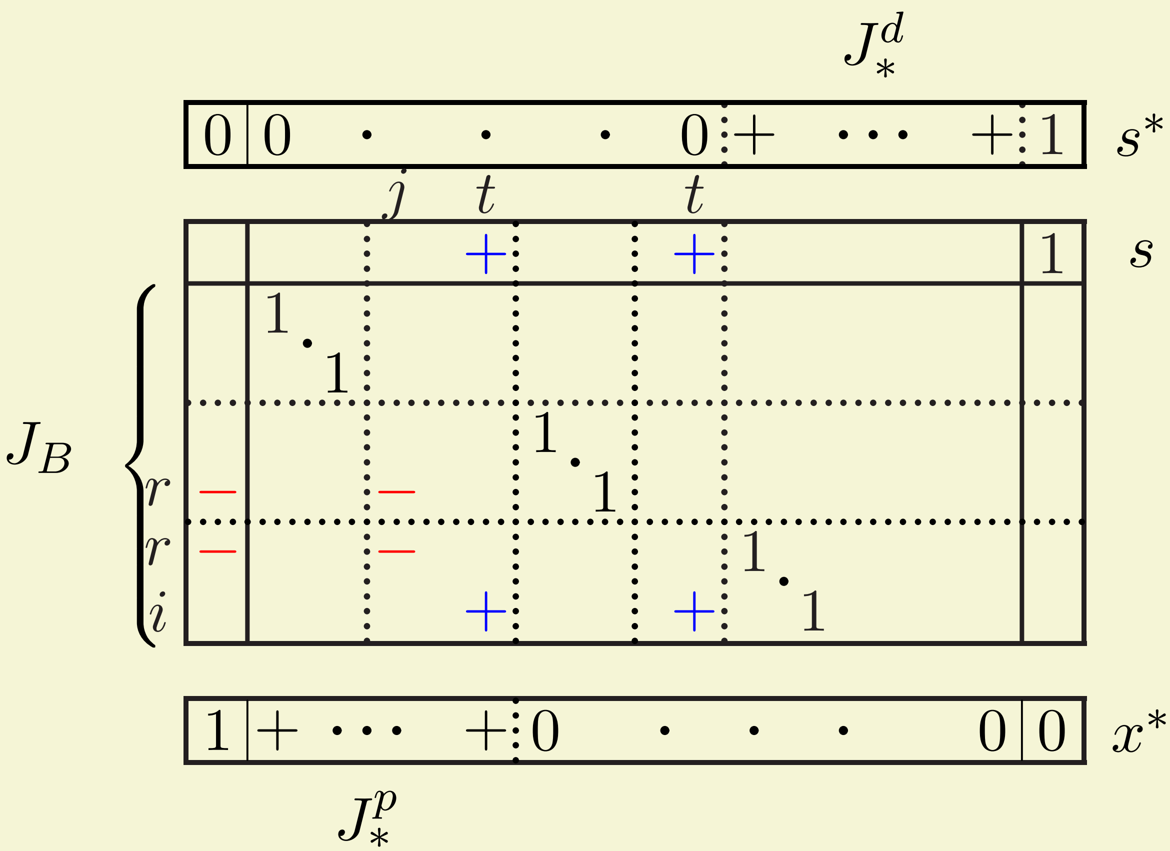
If $x \geq 0$, $s \geq 0$, then optimal basis is found.

Else reduce the basis

or the solution(s).

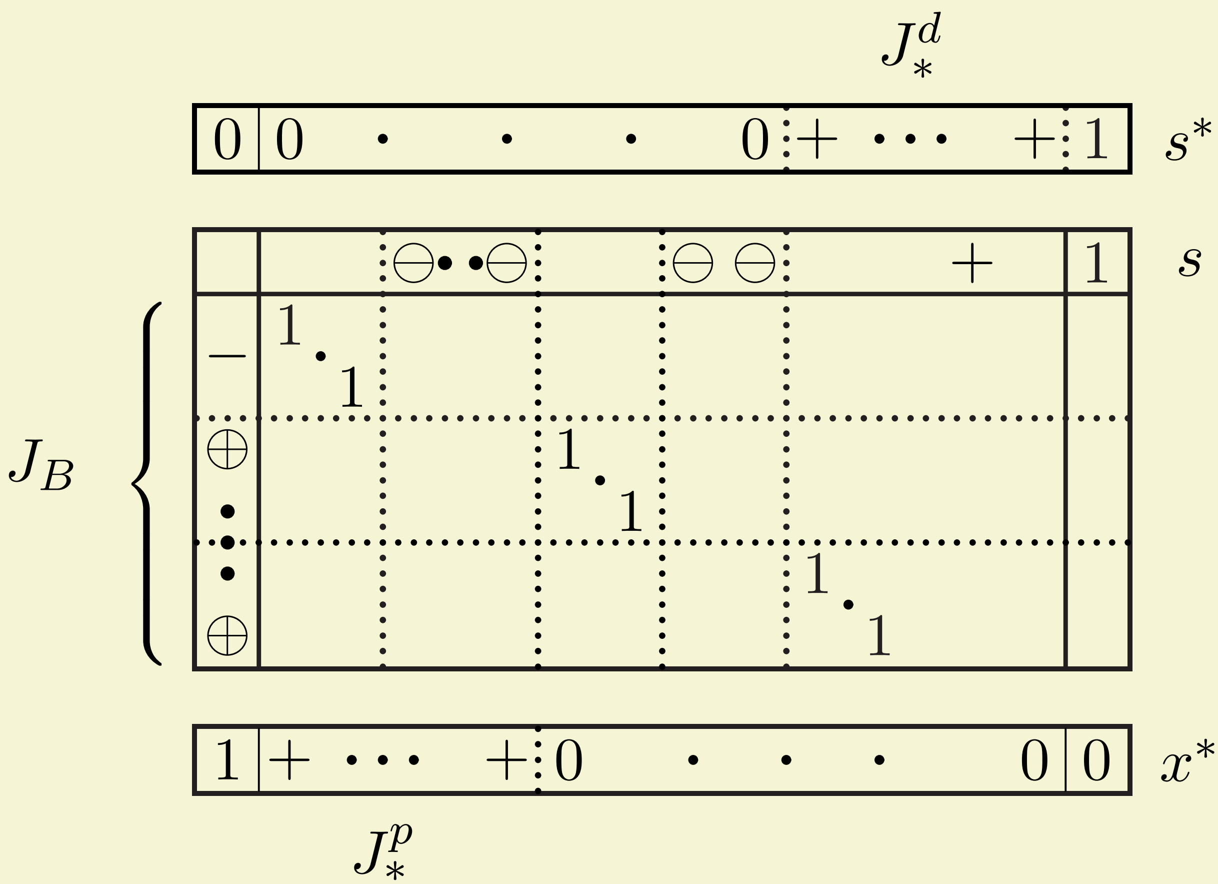
Reduce the basis

Either there is an $r \in J_B \cap \bar{J}_*^p$ with $x_r < 0$ and an $j \in J_*^p$ such that pivot (r, j) is admissible; or there is a $t \in \bar{J}_*^d$ with $s_t < 0$ and an $i \in J_B \cap \bar{J}_*^d$ such that pivot (i, t) is admissible.



Do pivot at (r, j) or at (i, t);
Update the data structure.
Restart the algorithm. Here neither x^* nor (y^*, s^*) change, only the basis.

Here $x_i \geq 0$ for all $i \in \bar{J}_B^p$, and
$s_j \geq 0$ for all $j \in \bar{J}_*^d$
but there is either an $i \in J_*^p$ such that $x_i < 0$,
or there is a $j \in J_*^d$ such that $s_j < 0$.
We eliminate either a coordinate of x^* of s^*.



(This structure is justified on the next sheet.)
Keeping optimality, we eliminate a coordinate of x^* or s^*.

Observations.

1. Because no desired admissible pivot exists, so we have
   - $-s_i \leq 0$ for $i \in \bar{J}_B \cap J_*^p$;
   - $x_i \geq 0$ for $i \in J_B \cap J_*^d$.
2. Although the current basis might be both primal and dual infeasible, the current objective value equals to the optimal one. Proof: Else, by taking convex combinations either better primal or better dual solution could be obtained.
3. Now it follows that $x_i = 0$ for $i \in J_B \cap J_*^d$; else, by taking convex combinations of the current and the optimal x would give an optimal x not complementaire with the optimal s^*.
4. Similarly, it follows that $s_i = 0$ for $i \in \bar{J}_B \cap J_*^p$; else, by taking convex combinations of the current and the optimal s would give an optimal s not complementaire with the optimal x^*.

In the case $s_j < 0$:
Let $\lambda := \min\left\{\frac{s_i^*}{s_i^* - s_i} \,|\, s_i < 0\right\} = \frac{s_t^*}{s_t^* - s_t}$.
Then we have $A^T y^* + s^* = c$ and $s^* \geq 0$ with
$(y^*, s^*) := \lambda(y, s) + (1-\lambda)(y^*, s^*)$ with $s_t^* = 0$.

Restart with the new updated data.
J_*^d has decreased!

One can operate analogously if an $x_i < 0$.

In the case $x_i < 0$:
Let $\lambda := \min\left\{\frac{x_i^*}{x_i^* - x_i} \,|\, x_i < 0\right\} = \frac{x_r^*}{x_r^* - x_r}$.
Then we have $Ax^* = b$ and $x^* \geq 0$ with
$x^* := \lambda x + (1-\lambda)x^*$ where $x_r^* = 0$.

Restart with the new updated data.
J_*^p has decreased!

Theorem 3 *Let us consider the LO problem*

$$\min\{c^T x \mid Ax = b, \quad x \geq 0\},$$

and its dual. Let a pair of optimal solutions x^*; (y^*, s^*) and any basis B be given. Then there is

- *a path of basis solutions initiated at B,*
- *leading to an optimal basis and*
- *connected by admissible pivots.*

The length of the path is at most n.

Proof: The algorithm presented above produces such an admissible pivot path.

At the step Reduce the basis: either $|J_B \cap \bar{J}_*^d|$ reduces by one, what might happen at, most $|J_B \cap J_*^d|$ times;

or $|J_B \cap J_*^p|$ increases by one, what might happen at most $|J_*^p|$ times.

Because $|J_B \cap J_*^d| + |J_*^p| \leq |J_*^d| + |J_*^p| \leq n$ we need at most n pivots to find an optimal basis. •

Note: The information that an optimal solution is known was heavily used. Basis identification in IPMs.

The algorithm does not give a strongly polynomial algorithm to solve the LO problem.